35. Congruences of the Eigenvalues of Hecke Operators

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Introduction. This note is a continuation of our previous note on the divisibility by 2 of the eigenvalues of Hecke operators [1]. We will omit the proofs of the theorems in this note. Details will appear in K. Hatada "On the eigenvalues of Hecke operators" [3].

§ 1. Let S_{w+2} be the space of cusp forms of weight $w+2$ on $SL(2, Z)$. Let λ_p be any eigenvalue of the Hecke operator $T(p)$ on S_{w+2} where p is a rational prime. In K. Hatada [1] we proved the following Theorem 1 and announced Theorem 2:

Theorem 1. *λ_p is divisible by 2 for any rational prime p and for any even weight $w+2$.*

Theorem 2. (i) *λ_p is divisible by 4 for any prime p with $p \equiv -1$* mod 4 *and for any even weight $w+2$.*

(ii) *$(\lambda_p - 2)$ is divisible by 4 for any prime p with $p \equiv +1$* mod 4 *and for any even weight $w+2$.*

Prof. J.-P. Serre sent us some experimental results, computed on a machine, which are proved by Theorems 2, 4 and 5 in this note.

Later he sent his conjectures compatible with the known results (see Remark 1 below), which are proved by Theorems 3 and 6. The author wishes to express his gratitude to Prof. Serre for his suggestions.

In § 1 of this note we give congruences for eigenvalues of the Hecke operators on S_{w+2}. They are Theorems 3–9.

Let λ_p be any eigenvalue of the $T(p)$ on S_{w+2}.

Theorem 3. $\lambda_p \equiv 1 + p \bmod 8$, *for any odd prime p and for any even weight $w+2$.*

Theorem 4. *λ_2 is divisible by 8 for any even weight $w+2$.*

Theorem 5. (i) *λ_2 is divisible by 16 for any weignt $w+2$ such that $w \equiv 0$* mod 4.

(ii) *λ_2 is divisible by 32 for any weight $w+2$ such that $w \equiv 0$* mod 4 *and $w \not\equiv 0$* mod 8.

Theorem 6. $\lambda_p \equiv 1 + p \bmod 3$ *for any rational prime p except for $p=3$ and any even weight $w+2$.*

Theorem 7. *λ_3 is divisible by 3 for any even weight $w+2$.*

Theorem 8. $\lambda_{11} \equiv 2 \bmod 5$ *for any even weight $w+2$.*

Theorem 9. $\lambda_{19} \equiv 0 \bmod 5$ *for any even weight $w+2$.*

Remark 1. Let tr $T(p)_{w+2}$ be the trace of the $T(p)$ on S_{w+2}. A few years ago Prof. Serre and Prof. Tate obtained the results that

$\mathrm{tr}\, T(p)_{w+2}=(1+p)\dim_C S_{w+2} \bmod 8$ for any prime p ($\neq 2$),

and that

$\mathrm{tr}\, T(p)_{w+2}\equiv(1+p)\dim_C S_{w+2} \bmod 3$ for any prime p ($\neq 3$).

Next Propositions 1, 2 and 3 are obtained by trace formula.

Proposition 1. $\mathrm{tr}\, T(5)_{w+2}\equiv 0 \bmod 5$ *for any even weight* $w+2$.

Proposition 2. $\mathrm{tr}\, T(7)_{w+2}\equiv 0 \bmod 7$ *for any even weight* $w+2$.

Proposition 3.

$$\mathrm{tr}\, T(11)_{w+2}\equiv\begin{cases}1 \bmod 11 & \text{if } w\equiv -1 \bmod 11\\ 0 \bmod 11 & \text{if } w\not\equiv -1 \bmod 11\end{cases}.$$

These propositions are obtained by Proposition 1 in M. Koike (Nagoya Math. Journal Vol. 56 (1973) 45–52).

§ 2. We consider in this § 2 congruences for eigenvalues of Hecke operators on cusp forms for some congruence subgroups $\subset SL(2, Z)$. The results given in this § 2 are not directly suggested by Prof. Serre, but they are related to the theorems in § 1 of this paper.

1) We set $S_{w+2}(\Gamma(2))=$the space of cusp forms of weight $w+2$ on $\Gamma(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2, Z)\,|\, a\equiv d\equiv 1 \bmod 2 \text{ and } b\equiv c\equiv 0 \bmod 2\right\}$. Let λ_p be any eigenvalue of the Hecke operator $T(p)$ on $S_{w+2}(\Gamma(2))$. Then we have

Theorem 10. $\lambda_p\equiv 1+p \bmod 4$ *for all odd primes* p *and for any even weight* $w+2$.

Let $S_{w+2}(\Gamma_0(N))$ be the space of cusp forms of weight $w+2$ on $\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\middle|\, N\,|\, c,\ ad-bc=1\right\}$.

2) Let λ_p be any eigenvalue of the $T(p)$ on $S_{w+2}(\Gamma_0(3))$.

Theorem 11. λ_p *is divisible by* 2 *for any odd prime* p (*at least except for* $p=3$) *and for any even weight* $w+2$.

Theorem 12. $\lambda_p\equiv 1+p \bmod 3$ *for any odd prime* p ($\neq 3$) *and for any even weight* $w+2$.

3) **Theorem 13.** *Any eigenvalue of the Hecke operators* $T(p)$ *on* $S_{w+2}(\Gamma_0(6))$ *is divisible by* 2 *for any rational prime* p *with* $p\equiv 1 \bmod 3$ *and for any even weight* $w+2$.

Remark 2. Similar results to Theorem 11 hold for $S_6(\Gamma_0(5))$ and $S_4(\Gamma_0(5))$.

§ 3. There are congruences of eigenvalues obtained from the ratio of the periods of primitive forms. (The basic reference is Manin [5].) Let $f\in S^0_{w+2}(\Gamma_0(N))$ be any primitive form. Set $f=\sum_{n=1}^{+\infty} a_n \exp 2\pi\, inz$. Set

$$R(l, g) = \mathrm{Re} \int_0^{i\infty} (f \mid [g])(z) z^l dz,$$
$$I(l, g) = \mathrm{Im} \int_0^{i\infty} (f \mid [g])(z) z^l dz.$$

We set $SL(2, \boldsymbol{Z}) = \bigcup_{j=1}^{m} \Gamma_0(N) g_j$, the left coset decomposition. We showed in [2] that both the ratio of $\{R(l, g_k)\}\ 0 \leqq l \leqq w,\ 1 \leqq k \leqq m$ and the ratio of $\{I(l, g_k)\}\ 0 \leqq l \leqq w,\ 1 \leqq k \leqq m$ are obtained by solving linear equations with coefficients in $\boldsymbol{Q}(a_1, a_2, a_3, \cdots)$. By direct computations of the ratio of $\{I(l, g_k)\}\ 0 \leqq l \leqq w,\ 1 \leqq k \leqq m$ and extending the coefficients theorem for $SL(2, \boldsymbol{Z})$ in Manin [5] to the congruence subgroup $\Gamma_0(N)$, we obtain some congruences of the coefficients of the f (see [2]).

Example 1. For $S_8(\Gamma_0(2))$, we have

$a_p \equiv 1 + p^7 \bmod 17$ for any odd prime p.

Example 2. For $S_{10}(\Gamma_0(2))$, we have

$a_p \equiv 1 + p^9 \bmod 31$ for any odd prime p.

Example 3. For $S_4(\Gamma_0(6))$, we have

$a_p \equiv 0 \bmod 2$ for any odd prime $p\ (\neq 3)$.

They are derived from

Lemma (K. Hatada [2] Lemma 12). *Let p be any prime with $p \nmid N$. Then there are rational integers $T_{j,l}(p)$ $(1 \leqq j \leqq m, 1 \leqq l \leqq w-1)$ which satisfy*

$$\int_0^{i\infty} F \mid T(p)(z) dz = (1 + p^{w+1}) \int_0^{i\infty} F(z) dz + \sum_{j=1}^{m} \sum_{l=1}^{w-1} T_{j,l}(p) \int_0^{i\infty} F \mid [g_j](z) z^l dz,$$

for all $F \in S_{w+2}(\Gamma_0(N))$. Here $w \geqq 2$.

For weight $w+2=2$ cases,

Example 4. For $S_2(\Gamma_0(11))$, we have

$a_p \equiv 1 + p \bmod 5$ for any odd prime p not dividing 5.

Example 5. For $S_2(\Gamma_0(17))$, we have

$a_p \equiv 1 + p \bmod 4$ for any odd prime $p\ (\neq 17)$.

Example 6. For $S_2(\Gamma_0(19))$, we have

$a_p \equiv 1 + p \bmod 3$ for any odd prime $p\ (\neq 19)$.

These Examples 4–6 are obtained by [4] 7.9 Theorem and 8.3 Computations of the tables, and are obtained by a different method (see [9], (7.6.19)).

They are analogue for $\tau(p) \equiv 1 + p^{11} \bmod 691$ where

$$q \prod_{n=1}^{+\infty} (1 - q^n)^{24} = \sum_{n=1}^{+\infty} \tau(n) q^n.$$

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