12. *On Riemannian Spaces Admitting a Family of Totally Umbilical Hypersurfaces. II.*

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§ 4. When orthogonal trajectories of the hypersurfaces $\sigma(x^\lambda)=$ const. are geodesics, (1.2) reduces to the form

$$\sigma_{\lambda;\mu} = \rho g_{\lambda\mu} + \eta\sigma_\lambda\sigma_\mu , \tag{4.1}$$

namely σ_λ is a torse-forming vector field. In this case, since v_λ is proportional to σ_λ and consequently $v_i = v_\lambda B_i^{\cdot\lambda} = 0$, (1.6) becomes

$$R_{jk} = B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\mu\nu} + \beta g_{jk} .$$

Thus we have

Theorem 4.1. When orthogonal trajectories of the totally umbilical hypersurfaces $\sigma(x^\lambda) =$ const. are geodesics, in order that the hypersurfaces $\sigma(x^\lambda) =$ const. are Einstein spaces, it is necessary and sufficient that the tensor $\Pi_{\lambda\mu}$ takes the form

$$\Pi_{\lambda\mu} = u g_{\lambda\mu} + \zeta_\lambda\sigma_\mu + \zeta_\mu\sigma_\lambda .$$

Cor. 1. If σ_λ is a torse-forming vector field and $\Pi_{\lambda\mu} = u g_{\lambda\mu} + \kappa\sigma_\lambda\sigma_\mu$, then the hypersurfaces $\sigma(x^\lambda) =$ const. are Einstein spaces.

Cor. 2. If an Einstein space admits a torse-forming vector field σ_λ, then the hypersurfaces $\sigma(x^\lambda) =$ const. are also Einstein spaces.

We consider next a conformally flat space admitting a torse-forming vector field.

Differentiating (4.1) and substituting the resulted equations in Ricci identities $\sigma_{\lambda;\mu\nu} - \sigma_{\lambda;\nu\mu} = -\sigma_\omega R^{\omega}_{\cdot\cdot\lambda\mu\nu}$, we have

$$-\sigma_\omega R^{\omega}_{\cdot\cdot\lambda\mu\nu} = (\rho_\nu - \rho\eta\sigma_\nu) g_{\lambda\mu} - (\rho_\mu - \rho\eta\sigma_\mu) g_{\lambda\nu} + \sigma_\lambda(\eta_\nu\sigma_\mu - \eta_\mu\sigma_\nu) . \tag{4.2}$$

Multiplying by σ^λ and summing for λ, we have

$$(\rho_\nu + \sigma^\lambda\sigma_\lambda\eta_\nu)\sigma_\mu - (\rho_\mu + \sigma^\lambda\sigma_\lambda\eta_\mu)\sigma_\nu = 0 ,$$

from which follows that $\rho_\nu + \sigma^\lambda\sigma_\lambda\eta_\nu$ is proportional to σ_ν, that is, $\sigma^\lambda\sigma_\lambda\eta_\nu = a\sigma_\nu - \rho_\nu$, where a is a certain scalar. On the other hand, multiplying (4.2) by $g^{\lambda\mu}$ and summing for λ and μ, we have

$$\begin{aligned} -\sigma_\omega R^{\omega}_{\cdot\cdot\nu} &= (n-1)(\rho_\nu - \rho\eta\sigma_\nu) + \sigma^\lambda\sigma_\lambda\eta_\nu - \sigma^\lambda\eta_\lambda\sigma_\nu \\ &= (n-2)\rho_\nu + \{a - (n-1)\rho\eta - \sigma^\lambda\eta_\lambda\}\sigma_\nu . \end{aligned}$$

Thus we obtain the equations of the form

$$\sigma_\omega \Pi^{\omega}_{\cdot\,\nu} = \rho_\nu + b\sigma_\nu\,,$$

where
$$b = \frac{1}{n-2}\left\{\frac{R}{2(n-1)} + a - (n-1)\rho\eta - \sigma^\lambda \eta_\lambda\right\}.$$

However, by virtue of $C^{\lambda}_{\cdot\,\mu\nu\omega} = 0$,

$$\begin{aligned} -\sigma_\omega R^{\omega}_{\cdot\,\lambda\mu\nu} &= \Pi_{\lambda\mu}\sigma_\nu - \Pi_{\lambda\nu}\sigma_\mu + g_{\lambda\mu}\sigma_\omega \Pi^{\omega}_{\cdot\,\nu} - g_{\lambda\nu}\sigma_\omega \Pi^{\omega}_{\cdot\,\mu} \\ &= (\Pi_{\lambda\mu} + bg_{\lambda\mu})\sigma_\nu - (\Pi_{\lambda\nu} + bg_{\lambda\nu})\sigma_\mu \\ &\quad + \rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}\,. \end{aligned}$$

Comparing with (4.2), we have

$$\left\{\Pi_{\lambda\mu} + (b+\rho\eta)g_{\lambda\mu} - \frac{1}{\sigma^\omega\sigma_\omega}\sigma_\lambda\rho_\mu\right\}\sigma_\nu - \left\{\Pi_{\lambda\nu} + (b+\rho\eta)g_{\lambda\nu} - \frac{1}{\sigma^\omega\sigma_\omega}\sigma_\lambda\rho_\nu\right\}\sigma_\mu = 0\,.$$

Consequently $\Pi_{\lambda\mu}$ takes the form

$$\Pi_{\lambda\mu} = ug_{\lambda\mu} + \zeta_\lambda\sigma_\mu + \zeta_\mu\sigma_\lambda\,.$$

Hence from the theorem 4.1 we know that the hypersurfaces $\sigma =$ const. are Einstein spaces. However since the totally umbilical surface in a conformally flat space is also conformally flat, the hypersurfaces $\sigma(x^\lambda) =$ const. are conformally flat and consequently of constant Riemann curvature. Thus we have

Theorem 4.2. When a conformally flat space admits a torse-forming vector field σ_λ, the hypersurfaces $\sigma(x^\lambda) =$ const. are of constant Riemann curvature ($n > 3$).

§ 5. The fundamental quadratic differential form of the space admitting a torse-forming vector field is given by the form[2)]

$$ds^2 = \rho(x^\lambda)^{-2} f_{ij}(x^k)dx^i dx^j + (dx^n)^2 \tag{5.1}$$
$$(i,\ j,\ k = 1,\ 2,\ \ldots,\ n-1)\,,$$

for a suitable coordinate system. Consequently the fundamental tensor becomes

$$g_{ij} = \rho^{-2} f_{ij}\,, \qquad g_{in} = 0\,, \qquad g_{nn} = 1\,,$$
$$g^{ij} = \rho^2 f^{ij}\,, \qquad g^{in} = 0\,, \qquad g^{nn} = 1\,,$$

where $f^{ij} f_{jk} = \delta^i_k$.

The fundamental tensor $\bar{g}_{ij}$ of the hypersurfaces $x^n =$ const. is equal to g_{ij}. If we denote the Christoffel symbols of the second kind, curvature tensor, Ricci tensor and scalar curvature with respect to $\bar{g}_{ij}$ by $\overline{\{^{i}_{jk}\}}$, $\bar{R}^{i}_{.jkl}$, $\bar{R}_{jk}$ and $\bar{R}$ respectively, we have from (5.1) the next equations:

$$(5.2)\quad \begin{cases} \begin{Bmatrix} n \\ nn \end{Bmatrix} = \begin{Bmatrix} i \\ nn \end{Bmatrix} = \begin{Bmatrix} n \\ i\,n \end{Bmatrix} = 0\,, \\ \begin{Bmatrix} n \\ jk \end{Bmatrix} = \rho_n g_{jk}\,, \qquad \begin{Bmatrix} i \\ j\,n \end{Bmatrix} = -\rho_n \delta^i_j\,, \\ \begin{Bmatrix} i \\ jk \end{Bmatrix} = \overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}} = \begin{Bmatrix} i \\ jk \end{Bmatrix}^* - \delta^i_j \rho_k - \delta^i_k \rho_j + f^{il} f_{jk} \rho_l\,, \end{cases}$$

where $\rho_\lambda = \dfrac{\partial \log \rho}{\partial x^\lambda}$ and $\begin{Bmatrix} i \\ jk \end{Bmatrix}^*$ are Christoffel symbols with respect to f_{ij};

$$(5.3)\quad \begin{cases} R^{n}_{.\,nnl} = R^{n}_{.\,nkl} = 0\,, \\ R^{n}_{.\,jnl} = \rho\left(\dfrac{1}{\rho}\right)_{nn} g_{jl}\,, \quad R^{i}_{.\,nnl} = -\rho\left(\dfrac{1}{\rho}\right)_{nn} \delta^i_l\,, \\ R^{n}_{.\,jkl} = \dfrac{\partial \rho_n}{\partial x^l} g_{jk} - \dfrac{\partial \rho_n}{\partial x^k} g_{jl}\,, \\ R^{i}_{.\,nkl} = -\dfrac{\partial \rho_n}{\partial x^l} \delta^i_k + \dfrac{\partial \rho_n}{\partial x^k} \delta^i_l\,, \\ R^{i}_{.\,jnl} = (\delta^i_l \delta^k_j - g^{ik} g_{lj}) \dfrac{\partial \rho_n}{\partial x^k}\,, \\ R^{i}_{.\,jkl} = \bar{R}^{i}_{.\,jkl} - \rho_n^2 (g_{jk}\delta^i_l - g_{jl}\delta^i_k)\,, \end{cases}$$

where $\left(\dfrac{1}{\rho}\right)_n = \dfrac{\partial}{\partial x^n}\left(\dfrac{1}{\rho}\right)$, $\left(\dfrac{1}{\rho}\right)_{nn} = \dfrac{\partial}{\partial x^n}\left(\dfrac{1}{\rho}\right)_n$;

$$(5.4)\quad \begin{cases} R_{nn} = -(n-1)\rho\left(\dfrac{1}{\rho}\right)_{nn}\,, \\ R_{jn} = (n-2)\dfrac{\partial \rho_n}{\partial x^j}\,, \\ R_{jk} = \bar{R}_{jk} - g_{jk}\left\{\rho\left(\dfrac{1}{\rho}\right)_{nn} + (n-2)(\rho_n)^2\right\}; \end{cases}$$

$$(5.5)\quad R = \bar{R} - (n-1)\left\{2\rho\left(\frac{1}{\rho}\right)_{nn} + (n-2)(\rho_n)^2\right\};$$

$$(5.6)\quad \begin{cases} \Pi_{nn} = \dfrac{\bar{R}}{2(n-1)(n-2)} + \rho\left(\dfrac{1}{\rho}\right)_{nn} - \dfrac{1}{2}(\rho_n)^2\,, \\ \Pi_{jn} = -\dfrac{\partial \rho_n}{\partial x^j}\,, \\ \Pi_{jk} = -\dfrac{1}{n-2}\left(\bar{R}_{jk} - \dfrac{\bar{R}}{2(n-1)} g_{jk}\right) + \dfrac{1}{2}(\rho_n)^2 g_{jk}\,; \end{cases}$$

$$(5.7)\quad \begin{cases} C^{n}_{\cdot nnl} = C^{n}_{\cdot nkl} = C^{n}_{\cdot jkl} = C^{i}_{\cdot nkl} = C^{i}_{\cdot jnl} = 0\,, \\ C^{n}_{\cdot jnl} = \dfrac{1}{n-2}\left(\bar{R}_{jl} - \dfrac{\bar{R}}{n-1}g_{jl}\right), \\ C^{i}_{\cdot nnl} = -\dfrac{1}{n-2}\left(\bar{R}^{i}_{\cdot l} - \dfrac{\bar{R}}{n-1}\delta^{i}_{l}\right), \\ C^{i}_{\cdot jkl} = \bar{R}^{i}_{\cdot jkl} - \dfrac{1}{n-2}\Big\{(\bar{R}^{i}_{\cdot l}g_{jk} - \bar{R}^{i}_{\cdot k}g_{jl} + \bar{R}_{jk}\delta^{i}_{l} - \bar{R}_{jl}\delta^{i}_{k}) \\ \qquad\qquad - \dfrac{1}{n-1}(\delta^{i}_{l}g_{jk} - \delta^{i}_{k}g_{jl})\Big\}\,; \end{cases}$$

$$(5.8)\quad \begin{cases} \Pi_{ni;j} - \Pi_{nj;i} = 0\,, \\ \Pi_{nn;i} - \Pi_{ni;n} = \dfrac{1}{2(n-1)(n-2)}\dfrac{\partial \bar{R}}{\partial x^{i}}\,, \\ \Pi_{ij;n} - \Pi_{in;j} = -\dfrac{\rho_{n}}{n-2}\left(\bar{R}_{ij} - \dfrac{\bar{R}}{n-1}g_{ij}\right) \\ \qquad - \dfrac{1}{n-2}\dfrac{\partial}{\partial x^{n}}\left(\bar{R}_{ij} - \dfrac{\bar{R}}{2(n-1)}g_{ij}\right) \\ \qquad + \dfrac{\partial}{\partial x^{n}}\left(\rho_{i\,|\,j} - \rho_{i}\rho_{j} + \dfrac{1}{2}g^{kl}\rho_{k}\rho_{l}g_{ij}\right), \\ \Pi_{ij;k} - \Pi_{ik;j} = -\dfrac{1}{n-2}\left\{\bar{R}_{ij} - \dfrac{\bar{R}}{2(n-1)}g_{ij}\right\}_{|k} \\ \qquad + \dfrac{1}{n-2}\left\{\bar{R}_{ik} - \dfrac{\bar{R}}{2(n-1)}g_{ik}\right\}_{|j}, \end{cases}$$

where $\rho_{i|j}$ are covariant derivatives with respect to $\bar{g}_{ij}$.

When $n > 3$, we get from (5.7) readily

Theorem 5.1. In order that the space admitting a torse-forming vector field σ_{λ} is conformally flat, it is necessary and sufficient that the hypersurfaces $\sigma(x^{\lambda}) = \text{const.}$ are of constant Riemann curvature ($n > 3$).

Now we consider (5.8). Without loss of generality, we may suppose that $\rho(x^{i}, x^{n}_{0}) = 1$ for some value x^{n}_{0} of x^{n}. Consequently the fundamental tensor of the hypersurface $x^{n} = x^{n}_{0}$ is given by f_{ij}. By a conformal transformation $g_{ij} = \rho^{-2}f_{ij}$, we have

$$(5.9)\quad \overline{\begin{Bmatrix} i \\ jk \end{Bmatrix}} = \begin{Bmatrix} i \\ jk \end{Bmatrix}^{*} - \delta^{i}_{j}\rho_{k} - \delta^{i}_{k}\rho_{j} + f^{il}\rho_{l}f_{jk}\,,$$

$$\bar{R}^{i}_{\cdot jkl} = R^{*i}_{\cdot jkl} + \rho^{*}_{jk}\delta^{i}_{l} - \rho^{*}_{jl}\delta^{i}_{k} + f_{jk}f^{im}\rho^{*}_{ml} - f_{jl}f^{im}\rho^{*}_{mk}\,,$$

where $R^{*i}_{\cdot jkl}$ is a curvature tensor with respect to f_{ij} and

$$\rho^{*}_{jk} = \frac{\partial \rho_j}{\partial x^k} - \left\{ {l \atop jk} \right\}^{*} \rho_l + \rho_j \rho_k - \frac{1}{2} f^{il} \rho_i \rho_l f_{jk} .$$

Hence we have

$$\bar{R}_{jk} = R^{*}_{jk} + (n-3)\rho^{*}_{jk} + f^{lm} \rho^{*}_{lm} f_{jk} , \tag{5.10}$$

$$\bar{R}\rho^{-2} = R^{*} + 2(n-2) f^{lm} \rho^{*}_{lm} , \tag{5.11}$$

where $R^{*}_{jk} = R^{*i}_{\cdot jki}$, $R^{*} = f^{jk} R^{*}_{jk}$. Furthermore, from (5.9) we have

$$\begin{aligned} \rho_{i|j} &\equiv \frac{\partial \rho_i}{\partial x^j} - \left\{ {\overline{l} \atop ij} \right\} \rho_l \\ &= \frac{\partial \rho_i}{\partial x^j} - \left(\left\{ {l \atop ij} \right\}^{*} - \delta^l_i \rho_j - \delta^l_j \rho_i + f^{lm} \rho_m f_{ij} \right) \rho_l , \end{aligned}$$

from which follows

$$\rho_{i|j} - \rho_i \rho_j + \frac{1}{2} g^{kl} \rho_k \rho_l g_{ij} = \rho^{*}_{ij} . \tag{5.12}$$

Let us suppose that V_n is comformally flat. Since the hyper surfaces $x^n = \text{const.}$ are Einstein spaces, we have

$$\bar{R}_{ij} - \frac{\bar{R}}{2(n-1)} g_{ij} = \frac{\bar{R}}{2(n-1)} g_{ij} = \frac{\bar{R}\rho^{-2}}{2(n-1)} f_{ij} .$$

Differentiating with respect to x^n,

$$\begin{aligned} \frac{\partial}{\partial x^n}\left(\bar{R}_{ij} - \frac{\bar{R}}{2(n-1)} g_{ij} \right) &= \frac{1}{2(n-1)} \frac{\partial}{\partial x^n} (\bar{R}\rho^{-2} f_{ij}) \\ &= (n-2) \frac{\partial}{\partial x^n} \left(\frac{1}{n-1} f^{lm} \rho^{*}_{lm} f_{ij} \right) . \end{aligned} \tag{5.13}$$

Substituting (5.12) and (5.13) in (5.8), we obtain

$$\Pi_{ij;n} - \Pi_{in;j} = \frac{\partial}{\partial x^n} \left(\rho^{*}_{ij} - \frac{1}{n-1} f^{lm} \rho^{*}_{lm} f_{ij} \right) = 0 ,$$

from which follows the equations of the form

$$\rho^{*}_{ij} - \frac{1}{n-1} f^{lm} \rho^{*}_{lm} f_{ij} = F(x^i) , \tag{5.14}$$

where $F(x^i)$ is a certain function of x^i. However, when $x^n = x^n_0$, by virtue of $\rho_i = 0$ and $\rho^{*}_{ij} = 0$, the left-hand member of (5.14) vanishes and the right-hand member does not involve x^n. Consequently (5.14) becomes

$$\rho^{*}_{ij} = \frac{1}{n-1} f^{lm} \rho^{*}_{lm} f_{ij} . \tag{5.15}$$

In this case, the vector field ρ_i of the hypersurfaces $x^n = x^n_0$ is a concircular one[1].

Moreover, we note that when $n > 3$ and the hypersurfaces $x^n = \text{const.}$ are all Einstein spaces, (5.15) may be reduced from (5.10).

In the case when $n = 3$, if V_3 is conformally flat, from (5.8) we know that $\bar{R}$ is a function of x^3 alone and consequently the hypersurfaces $x^3 = \text{const.}$ are of constant Riemann curvature. Furthermore we get the relations (5.15). Hence we have

Theorem 5.2. In order that V_3 which admits a torse-forming vector field and whose fundamental quadratic differential form is given by (5.1) is conformally flat, it is necessary and sufficient that the hypersurfaces $x^3 = \text{const.}$ are of constant Riemann curvature and the relations (5.15) hold, assuming $\rho(x^i, x^3_0) = 1$.

Furthermore from (5.8) we have

Theorem 5.3. In a space admitting a torse-forming vector field σ_λ, if the hypersurfaces $\sigma = \text{const.}$ are all Einstein spaces, then we have

$$\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = 0 \qquad (n > 3) .$$

References.

1) K. Yano: Concircular Geometry I. Proc. Imp. Acad. Tokyo, **16** (1940), 195–200.

2) K. Yano: On the Torse-forming Directions in Riemannian Spaces. Ibid. **20** (1944), 340–345.