26. *Generalisation of R. Baire's Theorem on Differential Equation* $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}f(x,y)=0$.

By Takashi KASUGA.

Department of Mathematics, Osaka University.

(Comm. by K. KUNUGI, M. J. A., March 12, 1951.)

Introduction.

In this paper, we shall denote by G a fixed open set in R^2 (Euclid plane defined by two coordinates x, y), by $f(x, y)$ a fixed continuous function defined everywhere in G, which has continuous f_y. (Functions will be always real-valued in this paper.)

We shall consider the partial differential equation

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}f(x, y)=0. \tag{1}$$

With (1), we shall associate the ordinary differential equation

$$\frac{dy}{dx}=f(x, y). \tag{2}$$

The curves representing the solutions of (2) which are prolonged up to the boundary of G on both sides and cannot be prolonged further, will be called characteristic curves (characteristics).

Through any point (x_0, y_0) in G, there passes one and only one characteristic curve $y=y(x, x_0, y_0)$.

(For the precise meanings of the propositions and their proofs cf. 1)).

We know[2)] that a continuous function $z(x, y)$ defined in G, is constant on each characteristic curve, if $z(x, y)$ has continuous $\partial z/\partial x$, $\partial z/\partial y$ or more generally is totally differentiable in G and satisfies (1) in G.

R. Baire showed in his thesis[3)] that, in the above, the assumption of continuity of $\partial z/\partial x$, $\partial z/\partial y$ or total differentiability of $z(x, y)$ is superfluous and, instead of them, only the existence of $\partial z/\partial x$, $\partial z/\partial y$ is sufficient.

We shall call it Baire's theorem.

In this paper we shall give a new proof of Baire's theorem by a method entirely different from his and somewhat generalise it.

We have not succeeded in proving this generalisation by Baire's original method.

We shall call it generalised Baire's theorem.

Generalised Baire's Theorem.

If a continuous function $z(x, y)$ defined in G has $\partial z/\partial x$, $\partial z/\partial y$ (not necessarily continuous), except at most at the points of an enumerable set, in G and satisfies (1) almost everywhere in G, then $z(x, y)$ is constant on each characteristic curve.

P. Montel, in 1913, gave a proof for an even more general theorem[4]. But he used a theorem on differential form $u(x, y)dx + v(x, y)dy$ of which the proof he has not published[5] and it seems to us that we need additional conditions on $f(x, y)$, for example, the existence of f_x, f_{yy} and continuity of f_{yy}, if we want to complete the proof of generalised Baire's theorem in his line of thought by known results[ ].

Proof of generalised Baire's theorem.

1. Let us denote by K the set of points (x_0, y_0) of G such that $z(x, y)$ is constant on the portion of each characteristic curve, contained in a fixed neighbourhood of (x_0, y_0).

We denote by F the set $G-K$. Evidently K is open and F is closed in G.

If F is empty, $z(x, y)$ is constant on each characteristic curve by the definition of K and the theorem is established.

Suppose therefore, if possible, that $F \neq 0$.

We denote by H the enumerable set consisting of the points at which $z(x, y)$ is not derivable with respect to x and with respect to y simultaneously.

If we denote by F_n, for each positive integer n, the set of points of G such that

$$|z(x+h, y)-z(x, y)| \leqq |h| n, \quad |z(x, y+k)-z(x, y)| \leqq |k| n,$$

whenever

$$|h|, |k| \leqq \frac{1}{n}, \quad (x+h, y), \quad (x, y+k) \in G,$$

then the sets F_n cover $G-H$ and each of the sets F_n is closed in G by dint of the continuity of $z(x, y)$. As we can easily see on account of the continuity of $z(x,y)$, the set F can contain no isolated point, that is, F is perfect in G.

Thus $F-H$ is not empty and of the second category as a G_δ in R^2.

Therefore there must exist a positive integer N and an open square Q: $|x-a| < L$, $|y-b| < L$ such that
$0 < L < 1/2N$, $\bar{Q} \subset G$, $(a, b) \in (F-H)\cdot Q \subset F_N$, (by $\bar{Q}$ we denote the closure of Q in R^2).

Then $$(a, b) \in F\cdot Q \subset F_N,$$

since the closure of $F-H$ in G is F by the perfectness of F in G and the enumerability of H.

2. By the continuity of $f(x, y)$, $f(x, y)$ is bounded in $\bar{Q}$. We denote by M the maximum of $|f(x,y)|$ in $\bar{Q}$ and let us write $l = L/M+1$.

We denote by η any number such that $|\eta-b|\leqq l$, then $\eta+lM\leqq b+L$, $\eta-lM\geqq b-L$.

Thus for any η there exists unique solution of (2) defined for $|x-a|<l$ which passes through (a, η) and lies in Q[7]. We denote it by $y=\varphi(x, \eta)$.

Hence, if we denote by Q_1 the domain defined by

$$\varphi(x, b-l)<y<\varphi(x, b+l), \quad |x-a|<l,$$

the curves $\varphi(x, \eta)$ fill up Q_1 simple-fold, when η takes all values in the open interval $|\eta-b|<l$[8], and $(a, b)\in Q_1\subset Q$.

We denote by Q_2 the open square $|\xi-a|<l$, $|\eta-b|<l$ in the (ξ, η)-plane.

Then $\varphi(\xi, \eta)$, $\partial\varphi/\partial\xi$, $\partial\varphi/\partial\eta$ exist and are continuous in Q_2 and

$$\frac{\partial\varphi}{\partial\eta} = \exp\left(\int_a^\xi f_y(x, \varphi(x, \eta))dx\right)>0 \text{ in } Q_2\text{[9]}.$$

Therefore, if we denote by A the one to one mapping of Q_2 on Q_1 defined by

$$x=\xi, \quad y=\varphi(\xi, \eta),$$

then the inverse mapping A^{-1} maps null sets in Q_1 to null sets in Q_2.

3. Clearly

$$(a, b)\in F\cdot Q_1\subset F\cdot Q\subset F_N.$$

We take any pair of points (x_1, y_1), (x_2, y_2) such that

$$(x_1, y_1)\in F\cdot Q_1, \quad (x_2, y_2)\in Q_1.$$

Then $\varphi(x_2, \eta_2)=\eta_2$ for an η_2 in the open interval $|\eta-b|<l$.

Now we denote by (x_3, y_3):

(Case I) the nearest point of F to (x_2, y_2) on the portion of the characteristic curve $y=\varphi(x, \eta_2)$ for $x_1\leqq x\leqq x_2$ or $x_2\leqq x\leqq x_1$, if it contains some points of F.

(Case II) the point $x_3=x_1$, $y_3=\varphi(x_1, \eta_2)$, if that portion contains no point of F.

Then in both cases, by the definition of F and continuity of $f(x, y)$,

$$z(x_3, y_3) = z(x_2, y_2)\,. \tag{3}$$

$\varphi(x, \eta_2) = y$ is a solution of (2), contained in Q and $|f(x, y)|<M$ in Q, thus

$$|y_2 - y_3| = |\varphi(x_2, \eta_2) - \varphi(x_3, \eta_2)| \leqq M|x_2 - x_3| \leqq M|x_1 - x_2|.$$

Hence

$$|y_1 - y_3| \leqq |y_1 - y_2| + |y_2 - y_3| \leqq |y_1 - y_2| + M|x_1 - x_2|. \tag{4}$$

We have

$$|z(x_1, y_3) - z(x_3, y_3)| \leqq N|x_1 - x_3|, \tag{5}$$

in the case I, as $(x_3, y_3) \in F\cdot Q_1 \subset F_N$, $(x_1, y_3) \in Q \subset G$, $|x_1 - x_3| < 2L < 1/N$, and in the case II, as $x_1 = x_3$.

Also we have

$$|z(x_1, y_1) - z(x_1, y_3)| \leqq N|y_1 - y_3, \tag{6}$$

as $(x_1, y_1) \in F\cdot Q_1 \subset F_N$, $|y_1 - y_3| < 2L < 1/N$, $(x_1, y_3) \in Q \subset G$.

By (3), (4), (5), and (6)

$$|z(x_1, y_1) - z(x_2, y_2)| = |z(x_1, y_1) - z(x_3, y_3)| \leqq N|x_1 - x_3| + N|y_1 - y_3| \leqq (NM+N)|x_1 - x_2| + N|y_1 - y_2|.$$

Hence

$$\overline{\lim_{(x_2, y_2)\to(x_1, y_1)}} \frac{|z(x_2, y_2) - z(x_1, y_1)|}{|x_1 - x_1| + |y_2 - y_1|} \leqq 2N + NM, \tag{7}$$

whenever

$$(x_2, y_2) \in Q_1, \quad (x_1, y_1) \in F\cdot Q_1.$$

4. From (7), $z(x, y)$, as a function in Q_1, is totally differentiable at almost all the points of $F\cdot Q_1$ by Stepanoff's theorem on almost everywhere total differentiablity[10]. Moreover $z(x, y)$ fulfills (1) almost everywhere in G and, as we have seen in 2, A^{-1} maps null sets in Q_1 to null sets in Q_2.

Hence, if we write $\zeta(\xi, \eta) = z(\xi, \varphi(\xi, \eta))$ for $(\xi, \eta) \in Q_2$,

$$\frac{\partial}{\partial\xi}\zeta(\xi, \eta) = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\,\frac{\partial\varphi}{\partial\xi} = \frac{\partial z}{\partial x} + f(x, y)\frac{\partial z}{\partial y} = 0$$

almost everywhere in $A^{-1}(F\cdot Q_1)$.

Also by (7), (if we write $x_1 = \xi_1$, $y_1 = \varphi(\xi_1, \eta_1)$)

$$\overline{\lim_{\xi\to\xi_1}} \frac{|\zeta(\xi, \eta_1) - \zeta(\xi_1, \eta_1)|}{|\xi - \xi_1|} \leqq \left(\overline{\lim_{(x, y)\to(x_1, y_1)}} \frac{|z(x, y) - z(x_1, y_1)|}{|x - x_1| + |y - y_1|}\right)$$

$$\times\left(\overline{\lim_{\xi\to\xi_1}} \frac{|\xi - \xi_1| + |\varphi(\xi, \eta_1) - \varphi(\xi_1, \eta_1)|}{|\xi - \xi_1|}\right) \leqq (2N + NM)\left(1 + \left|\frac{\partial\varphi}{\partial\xi}(\xi_1, \eta_1)\right|\right) =$$

$$(2N + NM)(1 + |f(x_1, y_1)|) \leqq (2N + NM)(1 + M)$$

for any $(\xi_1,\eta_1)\in A^{-1}(F\cdot Q_1)$.

Certainly $\frac{\partial}{\partial\xi}\zeta(\xi,\eta)=0$ in $Q_2-A^{-1}(F\cdot Q_1)=A^{-1}(K\cdot Q_1)$

as $z(x,y)$ is constant on a characteristic curve in some neighbourhood of its point (x,y). if $(x,y)\in K$.

Consequently $\frac{\partial\zeta}{\partial\xi}=0$ almost everywhere in Q_2,

and $\overline{\lim_{\xi\to\xi_1}}\frac{|\zeta(\xi,\eta_1)-\zeta(\xi_1,\eta_1)|}{|\xi-\xi_1|}\leq(1+M)(2N+NM)$ for any $(\xi_1,\eta_1)\in Q_2$.

5. Therefore by Fubini's Theorem, $\partial\zeta/\partial\xi$ vanishes almost everywhere in the interval $|\xi-a|<l$, as a function of ξ, for almost all η in the interval $|\eta-b|<l$. Moreover, $\zeta(\xi,\eta)$ is absolutely continuous. as a function of ξ, in interval $|\xi-a|<l$ for all η in the interval $|\eta-b|<l$.

Hence $\zeta(\xi,\eta)$ is constant, as a function of ξ, in the interval $|\xi-a|<l$, for almost all η in the interval $|\eta-b|<l$.

By the continuity of $z(x,y)$, accordingly, of $\zeta(\xi,\eta)$, the constancy of $\zeta(\xi,\eta)$ as a function of ξ, is established for all η in the interval $|\eta-b|<l$.

Thus $z(x,y)$ is constant on any characteristic curve in Q_1. This is however excluded, since $(a,b)\in F\cdot Q_1\neq 0$. We thus arrive at a contradiction and this completes the proof.

We shall treat the more general equation

$$\frac{\partial z}{\partial x}+f(x,y,z)\frac{\partial z}{\partial y}=g(x,y,z)$$

in a forthcoming paper.

I wish to express my gratitude for the kindness with which Prof. M. Nagumo and Prof. K. Kunugi have encouraged me through the course of this investigation.

References.

1) E. Kamke: Differentialgleichungen reller Funktionen, (1930) § 6, § 9, § 12.

2) E. Kamke: Loc. cit. p. 298.

3) R. Baire: Sur les fonctions de variable réelles, Ann. Mat. pura. appl. (3), 3, (1899) p. 100–p. 122.

4) P. Montel: Sur les differentielles totales et les fonctions monogènes, C. R. Acad. Sci. Paris, 156, (1913) p. 1820–p. 1822.

5) S. Saks: Théorie de l'intégrale, (1933) p. 243.

6) cf. 5).

7) cf. 1).

8) cf. 1),

9) E. Kamke: loc. cit. p. 155.

10) S. Saks: loc. cit. p. 238–p. 239.