108. On Some Representation Theorems in an Operator Algebra. II.

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3. **Application to a topological group.** We shall first prove Bochner's Theorem for a separable locally compact group by applying the Theorem 1 and its Remark 1, next prove the decomposition into irreducible factors for arbitray two sided unitary representation of a separable unimodular locally compact group, which has been called double unitary representation by R. Godement [8].

Theorem 3. *Let G be a separable locally compact group, and $\varphi(s)$ be a continuous positive define function on G. Then*

$$\varphi(s) = \int_R \chi(s, \lambda) d\sigma(\lambda) \tag{11}$$

where $\sigma(\lambda)$ is a suitable weight function which is an N-function of von Neumann [1], *and $\chi(s, \lambda)$ are elementary continuous positive definite functions for almost all λ in R. When $\varphi(s)$ is a central continuous p.d. (positive definite) function, these $\chi(s, \lambda)$ are also central elementary continuous p.d. functions.*

Proof. Let $\mathfrak{A}$ be L^1-algebra on G, then $\mathfrak{A}$ is a complete normed*-algebra with an approximate identity. We put

$$\varphi(x) = \int_G x(s)\varphi(s)ds \quad (x \in \mathfrak{A})$$

the integration being by Haar measure ds on G, then clearly $\varphi(x)$ is a state on $\mathfrak{A}$. Therefore, by the Theorem 1 and its Remark 1 there exists a system of pure states $\chi(x, \lambda)$, $\lambda \in N\sigma(\lambda)$-null set, such that

$$\varphi(x) = \int_R \chi(x, \lambda) d\sigma(\lambda).$$

By Riesz-Markoff-Kakutani's Theorem, there are elementary continuous p.d. functions $\chi(s, \lambda)$, $\lambda \in N$, such that

$$\chi(x, \lambda) = \int_G x(s)\chi(s, \lambda)ds.$$

For these $\chi(s, \lambda)$, we shall prove the relation (11). Let $\{V_n\}$ be an enumerable neighbourhoods system of the unite of G. For any $t \in G$

$$\lim_{n\to\infty} \int \chi(s, \lambda) C_{tV_n}(s)ds/|V_n| = \chi(t, \lambda) \quad \text{a.e. } \sigma(\lambda)$$

where $C_{tV_n}(s)$ be a caracteristic function of the set tV_n and $|V_n|$ be the volume for the Haar measure of G. Hence $\chi(t, \lambda)$ are $\sigma(\lambda)$-measurable for all $t \in G$. Let K be a subset of G and $a, b \in R$ (real number), we denote $M_{a,b,k}$ being a set of all $\lambda \in R-N$ such that $a \leq \chi(s, \lambda) \leq b$ for all s in K. Two topologies τ_1 and τ_2 on R are defined by the families of the subsets in $R\{M_{a,b,s} \mid a, b \in R, s \in G\}$ and $\{M_{a,b,k} \mid a, b \in R, K$ being runing over on the family of all compact set in $G\}$, respectively. Since G is separable, every Borel set in R_{τ_2} is also a Borel set in R_{τ_1}, and the $\sigma(\lambda)$-measureable. Moreover $\chi(s, \lambda)$ is continuous on the product topological space $G \times R_{\tau_2}$, hence $\chi(s, \lambda)$ is measurable for the product measure of G and R. By Fubini's Theorem, for all $x \in \mathfrak{A}$

$$
\begin{aligned}
\varphi(x) &= \int_R \int_G x(s)\chi(s, \lambda) ds\, d\sigma(\lambda) \\
&= \int_G \int_R x(s)\chi(s, \lambda) d\sigma(\lambda)\, ds.
\end{aligned}
\tag{12}
$$

Therefore we obtain the relation (11). On the case of central continuous p.d. function, we may prove in another paper with the decomposition of trace in C*-algebra.

Remark 2. As far as we know, the Bochner's Theorem for non-separable locally compact group has never been shown. R. Godement has given at a weak form (cf. [7]). Let G be a such group, Γ be a set of all elementary continuous p.d. functions and their weak limits. For any continuous p.d. function $\varphi(s)$, there exists a positive Radom measure $\mu(\cdot)$ such that

$$
\int_G x(s)\varphi(s) ds/\rho(s)^{1/2} = \int_\Gamma \int_G x(s)\chi(s) ds/\rho(s)^{1/2}\, d\mu(\chi) ds
$$

for all $x \in L^1$. The weak topology and compact open topology are coincide in Γ (cf. H. Yoshizawa [11]). Then, we have

$$
\varphi(s) = \int_\Gamma \chi(s) d\mu(\chi).
$$

But, on this case it is essential weak. For, as Godement has seen that Γ contains a p.d. function different from the elementary p.d. function. In his central group (cf. [10]), however, it may be held.

The uniform closure $R(G)$ of the collection of the operators of the form L_f (with $f \in L^1(G)$) is a C^*-algebra ($L_f g = f * g$, $g \in L^2(G)$ and its * being the convolution). We do not know that the complete relation between of the representations of $R(G)$ and G. But we know in L^1-algebra that there is a one-to-one correspondence between a continuous representation of $L^1(G)(=\mathfrak{A}$ say) and a continuous unitary representation of G, that is, $\{U_x, \mathfrak{H}\}$ be a continuous representation of $\mathfrak{A}$, then there exists a continuous unitary representation $\{U_s, \mathfrak{H}\}$ of G such that

$$U_x\xi = \int_G x(s)U_s\xi ds, \quad \xi \in \mathfrak{H} \tag{13}$$

where the integration is Banach space valued integral, and the converse correspondence be held by the same relation (13). Since the Theorem 2 can be applied for L^1-algebra[2], we have

Theorem 4. *Let G be a separable unimodular locally compact group. A normal two-sided continuous unitary representation of G is a directed integral of a system of irreducible such representations of G.*[3]

Proof. Give representation be $\{U_s, V_s, j, \mathfrak{H}\}$. For any $x \in \mathfrak{A}$ and any $\xi \in \mathfrak{H}$, we put

$$U_x\xi = \int x(s)U_s\xi ds, \quad V_x\xi = \int x(s^{-1})V_s\xi ds \tag{14}$$

where the integration being same way in (13). Then $\{U_x, V_x, j, \mathfrak{H}\}$ is a two-sided continuous representation of $\mathfrak{A}$. For, the conjugate linear transformation j is commute with the strong integration, that is, $(jU_xj\xi, \eta) = (j\eta, U_xj\xi) = \int\overline{x(s)}(j\eta, U_sj\xi)ds = \int\overline{x(s)}(jU_sj\xi, \eta)ds = \int\overline{x(s)}(V_s\xi, \eta)ds = (V_{x^*}\xi, \eta)$, hence $jU_xj = V_{x^*}$ and the other conditions are followed by $U_sV_t = V_tU_s$ and $V_{st} = V_tV_s$. In order to decompose $\{U_s, V_s, j, \mathfrak{H}\}$ into irreducible factors, first we shall prove that $\{U_x, V_y \mid x, y \in \mathfrak{A}\}' = \{U_s, V_t \mid s, t \in G\}'$. If $BU_s = U_sB$ and $BV_t = V_tB$, then since bounded linear operators are commute with the strong integration,

$$BU_x\xi = B\int_G x(s)U_s\xi ds = \int_G x(s)BU_s\xi ds = \int_G x(s)U_sB\xi ds$$

hence $BU_x = U_xB$ and by the same way $BV_x = V_xB$. The converse be possible to prove by the continuity of the representation $\{U_s, V_s, j, \mathfrak{H}\}$ and similar way above one. Thus, when we consider the decomposition of the Theorem 2 for L^1-algebra $\mathfrak{A}$, $\{U_x, V_x, j, \mathfrak{H}\}$ is a directed integral of a system of the irreducible two-sided representations $\{U_x(\lambda), V_x(\lambda), j(\lambda), \mathfrak{H}_\lambda\}$, $\lambda \in N\sigma(\lambda)$-null set. Since $\{U_x(\lambda), V_x(\lambda), j(\lambda), \mathfrak{H}_\lambda\}$ are continuous representations, there exist two-sided continuous unitary representations $\{U_s(\lambda), V_s'(\lambda), j(\lambda), \mathfrak{H}\}$[4] of G for

1) $\rho(s)$ is the measure factor of the Haar measure of G.

2) It is obvious by the same reason with the statement of Remark 1. The two-sided representation will be possible to define in an abstract * algebra, we shall discuss in another paper.

3) This theorem also holds for any such representation (being not always normal) onto a separable Hilbert space.

4) It can be proved by the same way with (14).

$\lambda \in N$ such that

$$U_x(\lambda)\xi_\lambda = \int_G x(s) U_s(\lambda)\xi_\lambda ds, \quad V_x(\lambda)\xi_\lambda = \int_G x(s^{-1}) V_s(\lambda)\xi_\lambda ds$$

for all $x \in \mathfrak{A}$ and $\xi_\lambda \in \mathfrak{H}_\lambda$. Then it remains to prove that the representation $\{U_s, V_s, j, \mathfrak{H}\}$ is a directed integral of the system of irreducible two-sided unitary representations $\{U_s(\lambda), V_s(\lambda), j(\lambda), \mathfrak{H}\}$. We have

$$(U_x\xi, \eta) = \int_G x(s)(U_s\xi, \eta)\, ds, \tag{15}$$

the left hand of (15) $= \int_R (U_x(\lambda)\xi_\lambda, \eta_\lambda) d\sigma(\lambda)$

$$= \int_R \int_G x(s)(U_s(\lambda)\xi_\lambda, \eta_\lambda) ds\, d\sigma(\lambda).$$

Now, we can apply the proof of theoren 3 for these functions $(U_s(\lambda)\xi_\lambda, \eta_\lambda)$ instesd of $\chi(s, \lambda)$ (cf. (12)) and hence can use Fubini's Theorem, so (15) is equal to

$$\int_G \int_R x(s)(U_s(\lambda)\xi_\lambda, \eta_\lambda) d\sigma(\lambda)\, ds.$$

As we have used sometimes (e.g. the equation (12))

$$\int_G x(s)(U_s\xi, \xi)\, ds\, d\sigma(\lambda) = \int_G \int_R x(s)(U_s(\lambda)\xi_\lambda, \xi_\lambda) d\sigma(\lambda)\, ds$$

for any $x(s)$ in $\mathfrak{A}$. Thus,

$$(U_s\xi, \xi) = \int_R (U_s(\lambda)\xi_\lambda, \xi_\lambda)\, d\sigma(\lambda)$$

for any $s \in G$. Since $\xi \in \mathfrak{H}$ is arbitrary, it completes the proof.

Add in proofs. From the equation (5) in the first paper I,P. 330, we have stated without proof that almost all $\{U_x(\lambda), \mathfrak{H}_\lambda\}$ are representations of $\mathfrak{A}$. Now we may prove this. Since $\mathfrak{A}$ is separable, there exists an enumerable dense self-adjoint subset $\mathfrak{A}_0 = \{X_n\}$ of $\mathfrak{A}$ such that $U_{x_n} \sim \sum U_{x_n}(\lambda)$ and $|||U_{x_n}(\lambda)||| \leqq |||U_{x_n}|||$, $U_{x_m x_n}(\lambda) = U_{x_m}(\lambda) U_{x_n}(\lambda)$, $U_{x_n}^*(\lambda) = U_{x_n}(\lambda)^*$ for $\lambda \bar{\in} N_{mn}$($\sigma(\lambda)$-null set). Put $N = \cup N_{mn}$. For any $x \in \mathfrak{A}$, there exists a sequence $\{x_n'\} \subset A_0$ such that $x_n' \to x$ in $\mathfrak{A}$. Since $|||U_{x_n'} - U_x||| \leqq ||x_n' - x|| \to 0$ and $|||U_{x_m'}(\lambda) - U_{x_n'}(\lambda)||| \leqq |||U_{x_m'} - U_{x_n'}|||$ for $\lambda \bar{\in} N$, we can find a bounded operator $A(\lambda)$ on $\mathfrak{H}_\lambda (\lambda \bar{\in} N)$ which is an uniform limit of $U_{x_n'}(\lambda)$. Hence $A(\lambda) = U_x(\lambda)$ for $\lambda \bar{\in} N$, and it may be proved $\{U_x(\lambda), \mathfrak{H}^\lambda\}$ $(\lambda \bar{\in} N)$ being representation.

Next, we have concluded from (8) that j_λ is our j-involution a.e. $\sigma(\lambda)$, there we had omitted the precise proof of the term of a.e. $\sigma(\lambda)$.

Throughout these papers I and II, we have described only summary notes. Their details will be descussed in more general from with other statements, it will appear elsewhere.

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