2. *On the Algebraic Construction of the Picard Variety.*

By Teruhisa MATSUSAKA.

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Introduction. In this paper, we shall show a purely algebraic construction of two Abelian Varieties called the "Picard Variety" and the "Albanese Variety" attached to a given complete normal Variety V^r for the universal domain of *arbitrary characteristic.* In the classical case, having the complex number field as the universal domain, in spite of the deep theories of Castelnuovo, Picard and Lefschetz, the distinction of the above two Abelian Varieties had been remained somewhat uncertain. Recently J. Igusa[1] has distinguished clearly these two Abelian Varieties and investigated the relations between them on the rigorous foundations of the theory of modern algebraic geometry and on the theory of harmonic differentials. On the duality theorem for Abelian Varieties and the theory of correspondences, we hope to study on some future occasions.

Let V^r be a normal projective Variety. W. L. Chow–v. d. Waerden's result on the associated-forms enable us to define algebraic families of positive cycles on V.[2),3)] According to this, there is a bunch $\mathfrak{F}$ in a projective space such that, every positive V-divisor of the given degree is in a one-to-one correspondence with the Point of a component of $\mathfrak{F}$. Let $\{X\}$ be the totality of V-divisors corresponding to a component U of $\mathfrak{F}$. We call $\{X\}$ as a (*maximal*) *algebraic family* and U the *associated-Variety* of $\{X\}$. The Point of U corresponding to a member X of it is called the *Chow-Point* of X. Let $G_a(V)$ be the group generated by all the V-divisors X–X', where X and X' belong to the same algebraic family. Any V-divisor belonging to $G_a(V)$ is called *algebraically equivalent to O.*

Let L be a generic linear Variety over a field of definition K for V such that $V. L$ is a Curve C. We shall say that C is a generic 1-section of V over K. C has no singular Point.[4)]

We shall use freely the results and terminologies in Weil's "Foundations of Algebraic Geometry", Amer. Math. Soc. Colloq., vol. 29, (1946) and "Variétés Abéliennes et Courbes Algébriques", Act. Sci. et Ind. (1948).

1) Forthcoming in the American Journal.

2) Cf. W.L. Chow-v.d. Waerden: "Zur algebraischen Geometrie IX". Math. Ann. 113.

3) Cf. W. L. Chow: "Algebraic system of positive cycles in an algebraic variety". Amer. Journ. vol. LXXII.

4) Cf. Y. Nakai: "On the section of an algebraic Variety by the generic hyperplane". Mem. Col. of Sci., Univ. Kyoto, Ser. A. XXVI. 1951.

§1. Let C be a generic 1-section of V over an algebraically closed field of definition k for V and K a common field of definition for C, for the Jacobian Variety J of C and for the canonical function φ of C containing k. Let U be the associated-Variety of a maximal algebraic family $\{X\}$ of positive V-divisors which is clearly defined over k and let x be a generic Point of U over K corresponding to X. Put

$$h(x) = S[\varphi(C.X)],$$

then h is a function on U with values in J. If Z is the graph of h, $Z \cap (J \times x')$ reduces to a Point for every Point x' of U. Let ξ be a generic Point of the projection B of Z on J over K. Put $(\xi \times U)$. $Z = \xi \times Z(\xi)$. If $\bar{C}$ is a generic 1-section of V over $K(\xi)$, then from the property of h we can show that every V-divisor corresponding to a Point of a component T of $Z(\xi)$ cuts out on $\bar{C}$ mutually linearly equivalent $\bar{C}$-divisor. As $\bar{C}$ is arbitrary, we conclude from this that *every member of $\{X\}$ corresponding to a Point of T is mutually linearly equivalent*[6].

As changing $\{X\}$ among the set of all the maximal algebraic families of positive V-divisors, *we can find $\{X\}$ with the following properties:*

(i) *B is the Abelian Subvariety of J, after a translation, if necessary,*

(ii) *if U' is the associated-Variety of a maximal algebraic family of positive V-divisors and B' the same as B, then B' is a Subvariety of B after a translation, if necessary.*

Let $\{X\}$ be such a maximal algebraic family and W its associated-Variety. Let U be the associated-Variety of a maximal algebraic family $\{Y\}$, X, Y be independent generic members of $\{X\}$, $\{Y\}$ over K and x, y the Chow-Points of them. $h(x \times y) = S[\varphi(C.X + C.Y)]$ is a function on $W \times U$ with values in J and let Z be its graph. Let ς be a generic Point of the projection B' of Z on J over K. Since $\{X\}$ satisfies (i) and (ii), every component of $Z(\varsigma)$ has the projection U on U and B' is the translation of B defined in (i) and (ii). From this and from what we have said above, the

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5) Cf. A. Weil's: "Variétés Abéliennes et Courbes Algébriques". Act. Sci. Ind. 1064, Théorémes 15-16, §5.

6) This follows from the following equivalence criteria, which was communicated to me by A. Weil.

"Let V be a normal projective Variety of dimension r. There is a finite numbers of (r-1)-dimensional Subvarieties $U_1, \dots, U_m$ of V with the following properties: let C be a generic 1-section of V over a common field of definition k for V and for every U_i, over which a given V-divisor X is rational, then if $C.X \sim O$, it holds $X \sim \Sigma\, a_i U_i$.

differences $\boldsymbol{Y}-\boldsymbol{Y}'$ of members of $\{\boldsymbol{Y}\}$, and *hence every member of* $G_a(\boldsymbol{V})$ *is linearly equivalent to the difference* $\boldsymbol{X}_1-\boldsymbol{X}_2$ *of two members of* $\{\boldsymbol{X}\}$ *and if it is rational over a field* K' *containing* k, *we can choose* $\boldsymbol{X}_i$ *in such a way that it is a generic member of* $\{\boldsymbol{X}\}$ *over* K' (i $=1$, 2).

§2. Let $\boldsymbol{X}$ be a generic member of $\{\boldsymbol{X}\}$ over k and $\boldsymbol{M}$ the Chow-Point of the associated-Variety $\boldsymbol{T}(\boldsymbol{X})$ of the complete linear system $|\boldsymbol{X}|$. Let $\boldsymbol{U}$ be the Locus of $\boldsymbol{M}$ over k. If $\boldsymbol{X}$, $\boldsymbol{X}_0$, $\boldsymbol{X}_1$ are three independent generic members of $\{\boldsymbol{X}\}$ over k, there is a generic member $\boldsymbol{X}_2$ of $\{\boldsymbol{X}\}$ over k such that $\boldsymbol{X}+\boldsymbol{X}_1-\boldsymbol{X}_0\sim\boldsymbol{X}_2$. Taking this into account, we can show that *$\boldsymbol{U}$ is birationally equivalent to an Abelian Variety $\boldsymbol{P}$*.[7] We define $\boldsymbol{P}$ as the *Picard Variety* of $\boldsymbol{V}$.

Denote by $G_l(\boldsymbol{V})$ the set of all the $\boldsymbol{V}$-divisors linearly equivalent to O. Then from the structure of $\boldsymbol{P}$, *the group* $G_a(V)/G_l(V)$ *is isomorphic with* $\boldsymbol{P}$. Moreover, the structure of $\boldsymbol{P}$ and the geometrical interpretations of it show that *there is a common field of definition* K *for* $\boldsymbol{V}$ *and* $\boldsymbol{P}$ *in such a way that if an element of* $G_a(\boldsymbol{V})/G_l(\boldsymbol{V})$ *has a representator, rational over a field* K' *containing* K, *then the Point of* $\boldsymbol{P}$ *corresponding to it is rational over* K'.[8]

Let $\boldsymbol{B}$ be the Variety defined in (i), (ii) of §1. We may assume, without loss of generality, that $\boldsymbol{B}$ is the Abelian Variety. Then $cat\ (\boldsymbol{B}) = cat\,(\boldsymbol{P})$ and there is a symmetric function on the Product $\boldsymbol{V}\times\ldots\times\boldsymbol{V}$ of q factors ($q=dim\ \boldsymbol{P}$) equal to $\boldsymbol{V}$ with values on $\boldsymbol{B}$. From this we conclude that *there is a symmetric function on* $\boldsymbol{V}\times\ldots\times\boldsymbol{V}$ *with values on* $\boldsymbol{P}$.

Let q' be the genus of a generic 1-section of $\boldsymbol{V}$ over k. Then every Abelian Variety generated by $\boldsymbol{V}$ cannot have the greater dimension than q'. This proves the *existence of the Abelian Variety* $\boldsymbol{A}$ *with the following property: there is a function* f *on* $\boldsymbol{V}$ *with values in* $\boldsymbol{A}$ *such that if* g *is a function on* $\boldsymbol{V}$ *with values in an Abelian Variety* $\boldsymbol{D}$, *we can find a homomorphism* λ *of* $\boldsymbol{A}$ *into* $\boldsymbol{D}$ *such that* $g=\lambda f+a$ *where* a *is a constant and* λ *is uniquely determined.* Assume that $\boldsymbol{A}$ has the minimal dimension among such Abelian Varieties. Then *$\boldsymbol{A}$ and $\boldsymbol{P}$ are determined uniquely up to an isomorphism by the class of birationally equivalent Varieties.* This $\boldsymbol{A}$ will be called as the "*Albanese Variety*" of $\boldsymbol{V}$ following A. Weil.

§3. *Let k be the smallest perfect field of definition for $\boldsymbol{V}$.* We regard this field k as the basic field. Assume that k has infinitely many elements. Then the Variety $\boldsymbol{U}$ defined in §2 derived from $\{\boldsymbol{X}\}$ satisfying the conditions of (i), (ii) of §1 is defined over k.

7) Cf. 5).

8) Prof. A. Weil kindly remarked the importance of this fact.

Let $\boldsymbol{B}$ be the Abelian Variety defined in (i), (ii) of § 1 in the Jacobian Variety $\boldsymbol{J}$ of a generic 1-section $\boldsymbol{C}$ of $\boldsymbol{V}$ over k. Taking Chow's recent results on Jacobian Varieties into account, and extending the algebraic family of Curves on $\boldsymbol{V}$ determined by a generic 1-section of $\boldsymbol{V}$ to the algebraic family of the Jacobian Varieties of them, we can find a projective Abelian Variety $\boldsymbol{B}'$ defined over k such that there are two generic Points x and y of $\boldsymbol{U}$ and $\boldsymbol{B}'$ over k such that $k(y) \subset k(x)$ and that $dim\ \boldsymbol{B}' = dim\ \boldsymbol{U}$. In this case, as $\boldsymbol{U}$ has the normal law of composition, *there is an Abelian Variety* $\boldsymbol{P}'$ *defined over* k, *birationally equivalent to* $\boldsymbol{U}$ *and hence to* $\boldsymbol{P}$, *immersed in a projective space.* The last assertion holds good, even when k is a finite field. It seems to be important to examine the first assertion in the case of a finite field.