27. *Probability-theoretic Investigations on Inheritance.* VII$_3$. *Non-Paternity Problems.*

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3. Sub-probabilities of proving non-paternity.

We have derived, in the preceding section, a formula (2.20) representing the whole probability of proving non-paternity by composing the sub-probabilities for various types of mothers. Though somewhat superfluous, we may think that the whole probability is composed of sub-probabilities concerning various kinds of mother-child combinations. We shall discuss, in the present section, such a decomposition.

We first consider the partial sum of probabilities of proving non-paternity, generally denoted by (2.2), corresponding to mother-child combination both consisting of homozygotes, necessarily of the same types. In view of the first relation (2.11), we get

$$\sum_{i=1}^{m} P(ii;\, ii) = \sum_{i=1}^{m} p_i^3 (1-p_i)^2 = S_3 - 2S_4 + S_5. \tag{3.1}$$

Next, the partial sum corresponding to those consisting of homozygotic mothers and of heterozygotic children is, in view of (2.13), given by

$$\begin{aligned}\sum_{i=1}^{m} \sum_{j\neq i} P(ii;\, ij) &= \sum_{i=1}^{m} p_i^2 (1-2S_2+S_3-p_i(1-p_i)^2)\\ &= S_2(1-2S_2+S_3)-(S_3-2S_4+S_5) = S_2-S_3-2S_2^2+2S_4+S_2S_3-S_5.\end{aligned} \tag{3.2}$$

The partial sum corresponding to mother-child combinations consisting of heterozygotic mothers and of homozygotic children is given by the sum of the first two terms of the left-hand side of (2.16). Each summand being symmetric with respect to suffices i and j, we can apply the general formula (1.7) and then obtain

$$\begin{aligned}\sum_{i,j}{}' (P(ij;\, ii)+P(ij;\, jj)) &= \sum_{i,j=1}^{m} P(i\mathring{j};\, ii) - \sum_{i=1}^{m} P(i\mathring{i};\, ii)\\ &= \sum_{i,j=1}^{m} p_i^2 p_j (1-p_i)^2 - \sum_{i=1}^{m} p_i^3 (1-p_i)^2\\ &= S_2-2S_3+S_4-(S_3-2S_4+S_5) = S_2-3S_3+3S_4-S_5.\end{aligned} \tag{3.3}$$

Here, the notation analogous to (2.19) has been used; namely,

$$\text{(3.4)}\qquad \begin{cases} P(\overset{\circ}{ii};ii)=[P(ij;ii)]^{p_j=p_i}, \\ P(\overset{\circ}{ij};ii)=P(ij;ii) \end{cases} \qquad (j\neq i).$$

The partial sum corresponding to those both consisting of heterozygotes of the same types is given by the sum of the third terms in (2.16) which yields

$$\text{(3.5)}\qquad \begin{aligned} \sum_{i,j}{}' P(ij;ij) &= \frac{1}{2}\Big(\sum_{i,j=1}^{m} P(\overset{\circ}{ij};\overset{\circ}{ij})-\sum_{i=1}^{m} P(\overset{\circ}{ii};\overset{\circ}{ii})\Big) \\ &= \frac{1}{2}\sum_{i,j=1}^{m} p_ip_j(p_i+p_j)(1-p_i-p_j)^2-\sum_{i=1}^{m} p_i^3(1-2p_i)^2 \\ &= S_2-2S_3-2S_2^2+S_4+3S_2S_3-(S_3-4S_4+4S_5) \\ &= S_2-3S_3-2S_2^2+5S_4+3S_2S_3-4S_5. \end{aligned}$$

Here, the notation analogous to (2.19) or (3.4) has been used; namely

$$\text{(3.6)}\qquad \begin{cases} P(\overset{\circ}{ii};\overset{\circ}{ii})=[P(ij;ij)]^{p_j=p_i}, \\ P(\overset{\circ}{ij};\overset{\circ}{ij})=P(ij;ij) \end{cases} \qquad (j\neq i).$$

It should again be noticed that

$$P(\overset{\circ}{ii};\overset{\circ}{ii})=2p_i^3(1-2p_i)^2\neq p_i^3(1-p_i)^2=P(ii;ii).$$

Last, the partial sum corresponding to those consisting of heterozygotes of different types, i.e., of types containing one gene alone in common, is given by the sum of (2.17) which becomes

$$\text{(3.7)}\qquad \begin{aligned} &\sum_{i,j}{}'\sum_{h\neq i,j}(P(ij;ih)+P(ij;jh)) \\ &=2\sum_{i,j}{}' p_ip_j(1-2S_2+S_3-p_i(1-p_i)^2-p_j(1-p_j)^2) \\ &\qquad -\sum_{i=1}^{m} p_i^2(1-2S_2+S_3-2p_i(1-p_i)^2) \\ &=1-2S_2+S_3-2(S_2-2S_3+S_4)-(S_2(1-2S_2+S_3)-2(S_3-2S_4+S_5)) \\ &=1-5S_2+7S_3+2S_2^2-6S_4-S_2S_3+2S_5. \end{aligned}$$

All the possible cases have thus been exhausted. It is a matter of course that the total sum of (3.1), (3.2), (3.3), (3.5) and (3.7) yields again the whole probability already given in (2.20).

In conclusion, if we sum up the quantities (2.14) over all i, we get the partial sum of probabilities with respect to homozygotic types of mothers:

$$\text{(3.8)}\qquad \sum_{i=1}^{m} P(ii)=S_2(1-2S_2+S_3),$$

which is evidently equal to the sum of (3.1) and (3.2). Similarly, that with respective to heterozygotic types of mothers is given by

$$\text{(3.9)}\qquad \sum_{i,j}{}' P(ij)=1-3S_2+S_3+2S_4+2S_2S_3-3S_5,$$

which is equal to the sum of (3.3), (3.5) and (3.7).

It may be noticed that the right-hand sides of (3.1), (3.2), (3.3), (3.5), (3.7), (3.8), (3.9) and also of (2.20) are not homogeneous with respect to indices in S's. If it would be desired to obtain homogeneous expressions, we have only to remember the second identity (1.3), i.e., $S_1=1$. The expressions homogeneous in S's of weight five, that is, homogeneous with respect to p's of degree five, will then be obtained by multiplying each term by a suitable power of S_1.

It will further be noticed that the sum of coefficients in every expression is equal to zero. This is a fact quite reasonable, explained as follows. Indeed, if the distribution-probabilities $p_i (i=1, \ldots, m)$ all vanish but one alone among them which must, of course, be equal to unity, then, all the power-sums with suffices $\nu \geqq 1$ become equal to unity, i.e., $S_\nu=1$ $(\nu=1, 2, \ldots)$. On the other hand, in such a degenerate case where only one gene is quite really existent, the proof of non-paternity is evidently impossible so that every expression in question must then vanish out. This explains the reason of the above mentioned fact.

The results obtained above, those contained in the previous table inclusive, may be listed as follows.

Mother	Child	Prob. of mother-child comb.	Deniable man	Freq. of deniable man
A_{ii}	A_{ii}	p_i^3	A_{hk} $(h, k \neq i)$	$(1-p_i)^2$
	A_{ij} $(j \neq i)$	$p_i^2 p_j$	A_{hk} $(h, k \neq j)$	$(1-p_j)^2$
A_{ij} $(i \neq j)$	A_{ii}	$p_i^2 p_j$	A_{hk} $(h, k \neq i)$	$(1-p_i)^2$
	A_{jj}	$p_i p_j^2$	A_{hk} $(h, k \neq j)$	$(1-p_j)^2$
	A_{ij}	$p_i p_j (p_i + p_i)$	A_{hk} $(h, k \neq i, j)$	$(1-p_i-p_j)^2$
	A_{ih} or A_{jh} $(h \neq i, j)$	$p_i p_j p_h$	A_{kl} $(k, l \neq h)$	$(1-p_h)^2$

Sub-prob. for each mother-child comb.	Prob. of proving non-pat. on each type of mother	Partial sum of prob. corresp. to each of mother-child comb.
$p_i^3(1-p_i)^2$	$p_i(1-2S_2+S_3)$	$S_3-2S_4+S_5$
$p_i^2 p_j (1-p_j)^2$		$S_2-S_3-2S_2^2+2S_4+S_2S_3-S_5$
$p_i^2 p_j (1-p_i)^2$	$p_i p_j (2(1-2S_2+S_3)$ $-4p_i p_j + 3p_i p_j (p_i+p_j))$	$\left.\right\}$ $S_2-3S_3+3S_4-S_5$
$p_i p_j^2 (1-p_j)^2$		
$p_i p_j (p_i+p_j)(1-p_i-p_j)^2$		$S_2-3S_3-2S_2^2+5S_4+S_2S_3-4S_5$
$p_i p_j p_h (1-p_h)^2$		$1-5S_2+7S_3+2S_2^2-6S_4$ $-S_2S_3+2S_5$

$$P=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5$$

—To be continued—