36. *Note on S-Groups*

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We want, in the present note, to show that some theorems concerning finite soluble groups can be extended to soluble groups admitting the maximal condition for subgroups, which we call, after K.A. Hirsch, S-groups. The proofs are based on a fundamental theorem of K.A. Hirsch, which states that the structural property of an S-group G occurs in a certain finite factor group of G, and are reduced to those on finite soluble groups.

(0)[1] We can formulate the fundamental theorem of K.A. Hirsch in an a little stronger (though formally) form :

Let G be an S-group. Let H be a normal subgroup of G. Assume that H is neither strongly soluble[2] nor nilpotent nor abelian. Then there exists a family of normal subgroups $\{N_\alpha\}$ of G satisfying the following conditions ; (1) each N_α is contained in H and has the finite index with respect to H, (2) each H/N_α is neither strongly soluble nor nilpotent nor abelian and (3) $\bigcap_\alpha N_\alpha = 1$.

(1)[3] [H. Wielandt] Let G be an S-group. Let $\Phi(G)$ and $D(G)$ be the Frattini subgroup and the commutator subgroup of G respectively. If $\Phi(G)$ contains $D(G)$, then G is nilpotent.

Proof. If G is not nilpotent, G contains a normal subgroup N such that G/N is finite, is not nilpotent and satisfies the same condition as G, which contradicts with a theorem of H. Wielandt.

(2)[*] The Frattini subgroup $\Phi(G)$ of an S-group G is nilpotent.

Proof. We may assume that $\Phi(G)$ is finite. In fact, if $\Phi(G)$ is not nilpotent, then $\Phi(G)$ contains a normal subgroup N such that N is normal also in G and $\Phi(G)/N$ is finite and not nilpotent. So, let us assume that $\Phi(G)$ is finite. Then we can take a normal subgroup H such that $H \cap \Phi = 1$ and G/H is finite. Thus the assertion is reduced to the case of finite groups, for which the assertion is well known.

(3)[4] Let G be a nilpotent S-group. Let H be a subgroup of G such that $D(H)$ is contained properly in $D(G)$. Then $D(\Phi(G)\cdot H)$ is also contained properly in $D(G)$.

Proof. First we assume that the assertion is valid for normal subgroups and that H is not normal in G. Let L be the least normal subgroup of G containing H. If $D(L)\neq D(G)$, then $D(\Phi(G)\cdot L)\neq D(G)$ by assumption. Then $D(\Phi(G)\cdot H)\neq D(G)$, a fortiori. If $D(L)=D(G)$,

*) This theorem has been obtaiend by K.A. Hirsh independently (unpublished).

then $HD(L)=HD(G)$ is normal in G. Since $D(L)\leqq\Phi(L)$, $HD(G)=HD(L)\leqq H\cdot\Phi(L)\neq 1$, which contradicts with the minimality of L.

Thus we have merely to treat the case when H is normal in G. Suppose then that the normal subgroup H is not abelian. Consider $G/D(H)$. Since $H/D(H)$ is abelian and $G/D(H)$ is not abelian, $D(\Phi(G)\cdot H)\neq D(G)$ by assumption.

Finally we consider an abelian normal subgroups H and assume that the assertion is valid for abelian normal subgroups with finite orders. Let us take a family of normal subgroups $\{N_\alpha\}$ of H satisfying the conditions in (0). If G/N_α is not abelian, then, since H/N_α is finite, $D(\Phi(G)\cdot H)\neq D(G)$ by assumption. If G/N_α is abelian for every α, then $D(G)=1$, which contradicts with $D(G)\neq D(H)$. Therefore we may assume that H is finite. Then there exists a normal subgroup N of G such that $N\cap H=1$ and G/N is finite and not abelian. Thus the assertion is reduced to the case of finite groups, for which the assertion is known.

Remark. In this way many results of finite soluble groups, apart from those of purely Sylowian character, can be extended to those of S-groups.

References

1) K.A. Hirsch: On infinite soluble groups (III), Proc. London Math. Soc., (2) **49**, 184-194 (1946); (IV), J. London Math. Soc., **27**, 81-85 (1952).

2) We call an S-group G strongly soluble if G possesses a weak composition series which is a weak principal series. For this see G. Zappa, Sui gruppi di Hirsch, supersolubili (I), (II), Rend. Sem. Mat. Univ. Padova, **12**, 1-11, 62-80 (1941).

3) H. Wielandt: Eine Kennzeichnung der direkten Produkte von p-Gruppen, Math. Zeit., **41**, 281-282 (1936).

4) N. Itô: On a theorem of L. Rédei and J. Szép concerning p-groups, Acta Sci. Math., **14**, 186-187 (1952).