181. *Continuity of Stochastic Processes on Metric Spaces*

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1. After A. N. Kolmogorov had presented the continuity condition of stochastic processes ([5]), several generalizations have been considered (e.g. [1]-[4]). But H. Cramér's idea in [1] permits us to obtain the continuity conditions in the more general situations; Let $\{x(t,\omega)\,;\,t\in S\}$ be stochastic processes, based on a probability space $(\Omega,\mathcal{B},P)$, of which parameter t runs over a compact metric space (S,d), and of which value is taken in a complete metric space (M,r). Here their metrics are d and r, respectively. Denote by $N(\varepsilon)$ the minimal number of ε-net of the space S.*) Then we establish the followings:

Theorem 1. *Suppose that*

(1) $$P[r(x(t),x(s))\geq g(d(t,s))]\leq q(d(t,s)),$$

where $g(h)$ and $q(h)$ are even, non-decreasing functions in $h>0$, and that

(2) $$\sum_{n=1}^{\infty} g(2^{-n})<\infty,\qquad \sum_{n=1}^{\infty} N^2(2^{-n-1})\cdot q(2^{-n+2})<\infty.$$

Then the stochastic processes have continuous version.

Theorem 2. *Suppose* (1) *above and that*

(3) $$\sum_{k=1}^{\infty} g(2^{-n-k})<C\cdot g(2^{-n}),\qquad \sum_{n=1}^{\infty} N^2(2^{-n-1})\cdot q(2^{-n+2})<\infty,$$

and

(4) $$g(4h)<C'\cdot g(h)\qquad \textit{for sufficiently small } h,$$

where n is any positive integer, and C and C' are some positive constants. Then the stochastic processes have g-Hölder continuous version.

2. By A_n, we denote the elements of 2^{-n}-*net*; $A_n=\{t_n^k\,;\,k=1,2,\cdots,N(2^{-n})\}$, $n=1,2,3,\cdots$, and we set $D=\bigcup_{n=1}^{\infty}A_n$. By F, we define the space of all M-valued, non-random functions, and by F_n the elements of F such that

$$F_n=\{f(t)\,;\,r(f(x),f(y))\leqq g(d(x,y)),\ \text{for}\ (x,y),x\in A_n,y\in A_{n+1}\ \text{and}\ d(x,y)\leqq 2^{-n+2}\},$$

where $g(h)$ is one cited in (1). Further, we set $U_n=\bigcap_{j=n}^{\infty}F_j$, and

*) $\log N(\varepsilon)$ is called ε-entropy of the space S.

$U=\bigcup_{n=1}^{\infty} U_n$. The function $f_D(t)$ is denoted as the restriction of $f(t) \in F$ to D, and $f_D(t+)$ means the limiting value of $\{f_D(t_n), t_n \in A_n, d(t, t_n) \leq 2^{-n+1}, n=1, 2, \cdots\}$, if it exists. Then we have;

Lemma 1. *If $f(t) \in U, f_D(t+)$ exists uniquely and independently of the sequence $\{t_n\}$.*

Proof. For any positive $\varepsilon>0$, there exists a number n_0 such that $\sum_{k=n_0}^{\infty} g(2^{-k})<\varepsilon$. For a sequence $\{t_n \in A_n, d(t, t_n) \leq 2^{-n+1}, n=1, 2, 3, \cdots\}$, and for any $p>q>1+\max(N, n_0)$ where N is the smallest number satisfying $f(t) \in U_N$**), we estimate

$$\begin{aligned} r(f_D(t_q), f_D(t_p)) &\leqq \sum_{l=q}^{p-1} r(f_D(t_l), f_D(t_{l+1})) \leqq \sum_{l=q}^{p-1} g(d(t_l, t_{l+1})) \\ &\leqq \sum_{l=q}^{\infty} g(2^{-l+2}) \leqq \sum_{l=n_0}^{\infty} g(2^{-l})<\varepsilon, \end{aligned}$$

since $d(t_l, t_{l+1}) \leq d(t_l, t)+d(t, t_{l+1}) \leq 2^{-l+1}+2^{-l} \leq 2^{-l+2}$, and $t_l \in A_l, t_{l+1} \in A_{l+1}$. Thus, we have

$$\lim_{l \to \infty} r(f_D(t_l), f_0)=0.$$

This f_0 does not depend on the sequence; In fact, if we have a different value f_0' for another sequence $\{t_n' ; t_n' \in A_n, d(t, t_n') \leq 2^{-n+1}, n=1, 2, 3, \cdots\}$, we can observe that, since $d(t_n, t_{n+1}') \leq d(t_n, t)+d(t, t_{n+1}') \leq 2^{-n+1}+2^{-n} \leq 2^{-n+2}$,

$$\begin{aligned} r(f_0, f_0') &\leq r(f_0, f_D(t_n))+r(f_D(t_n), f_D(t_{n+1}'))+r(f_D(t_{n+1}'), f_0') \\ &\leq 2\varepsilon+g(2^{-n+2}), \end{aligned}$$

for any $\varepsilon>0$ and for sufficiently large n. This proves the Lemma.

Lemma 2. *For $f(t) \in U$, we set $h(t)=f_D(t+)$. Then $h(t)$ is continuous in t.*

Proof. We shall show by contradiction; Assume that there exists a sequence $\{x_n ; x_n \in S\}$ converging to t such that $\lim_{n \to \infty} r(h(x_n), h(t)) \neq 0$. For each integer m, we can find a point x_{n_m} in the sequence $\{x_n\}$ satisfying $d(t, x_{n_m}) \leq 2^{-m}$, and further for each x_{n_m}, there exists a sequence $\{y_q(n_m) ; y_q(n_m) \in A_q, d(x_{n_m}, y_q(n_m)) \leq 2^{-q}, q=1, 2, \cdots\}$, for which we have $\lim_{q \to \infty} r(f_D(y_q(n_m), h(x_{n_m}))=0$, due to Lemma 1. Since

$$d(t, y_m(n_m)) \leq d(t, x_{n_m})+d(x_{n_m}, y_m(n_m)) \leq 2^{-m}+2^{-m}=2^{-m+1},$$

and $y_m(n_m) \in A_m$, we have $\lim_{m \to \infty} r(f_D(y_m(n_m)), h(t))=0$. Further we estimate

$$\begin{aligned} r(h(t), h(x_{n_m})) &\leqq r(h(t), f_D(y_m(n_m)))+r(f_D(y_m(n_m)), h(x_{n_m})) \\ &\leqq r(h(t), f_D(y_m(n_m)))+\sum_{q=m}^{Q-1} r(f_D(y_p(n_m)), f_D(y_{q+1}(n_m))) \\ &\quad +r(f_D(y_Q(n_m)), h(x_{n_m})). \end{aligned}$$

Here, by making the integer Q so large, we have for any $\varepsilon>0$,

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**) We shall not repeat below the indication of the number to which class of U a function $f(t)$ belongs, when it is clear in the context.

$$\sum_{q=m}^{Q-1} r(f_D(y_q(n_m)), f_D(y_{q+1}(n_m))) + r(f_D(y_Q(n_m)), h(x_{n_m}))$$
$$\leqq \sum_{q=m}^{\infty} r(f_D(y_q(n_m)), f_D(y_{q+1}(n_m))) + \varepsilon.$$

Further, since $d(y_q(n_m), y_{q+1}(n_m)) \leq d(y_q(n_m), x_{n_m}) + d(x_{n_m}, y_{q+1}(n_m)) \leq 2^{-q} + 2^{-q-1} \leqq 2^{-q+2}$, it holds

$$\sum_{q=m}^{\infty} r(f_D(y_q(n_m)), f_D(y_{q+1}(n_m))) \leqq \sum_{q=m}^{\infty} g(2^{-q+2}).$$

As a result, we have

$$r(h(t), h(x_{n_m})) \leqq r(h(t), f_D(y_m(n_m))) + \sum_{k=m-2}^{\infty} g(2^{-k}),$$

which implies, for sufficiently large m, a contradiction to the hypothesis above.

Lemma 3. *Let $\{x(t, \omega); t \in S\}$ be stochastic processes satisfying the conditions* (1) *and* (2). *Then*

$$P[x(t, \omega) \in U] = 1.$$

Proof. For the complement of F_n, F_n^c we have

$$P[x(t, \omega) \in F_n^c] \leqq P[\max_{\substack{(t,s) \in A_n \times A_{n+1} \\ d(t,s) \leqq 2^{-n+2}}} r(x(t), x(s)) > g(d(t, s))]$$
$$\leqq \sum_{\substack{(t,s) \in A_n \times A_{n+1} \\ d(t,s) \leq 2^{-n+2}}} P[r(x(t), x(s) > g(d(t, s))]$$
$$\leqq N(2^{-n}) \cdot N(2^{-n-1}) \cdot q(2^{-n+2}).$$

Since we have $U_n^c = \bigcup_{j=n}^{\infty} F_j^c$, and

$$P[x(t, \omega) \in U_n^c] \leqq \sum_{j=n}^{\infty} P[x(t, \omega) \in F_j^c]$$
$$\leqq \sum_{j=n}^{\infty} N^2(2^{-n-1}) \cdot q(2^{-n+2}),$$

we obtain

$$P[x(t, \omega) \in U^c] = \lim_{n \to \infty} P[x(t, \omega) \in U_n^c] = 0,$$
$$P[x(t, \omega) \in U] = 0.$$

3. Proof of Theorem 1. For each t in S, we define ω-sets, V_t and W_t respectively as follows;

$$V_t = \{\omega; x(t, \omega) = x_D(t+, \omega)\},$$
$$W_t = \{\omega; x(t, \omega) \neq x_D(t+, \omega)\}.$$

We shall prove that

$$P[V_t] = 1, \quad \text{and} \quad P[W_t] = 0.$$

For $t \in S$, we choose a sequence $\{t_k; t_k \in A_k, d(t, t_k) \leq 2^{-k+1}, d(t, t_{k+1}) \leq d(t, t_k); k = 1, 2, \cdots\}$. Then, due to Lemmas 1 and 3, we have

$$P[r(x(t), x(t_k)) \geqq g(d(t, t_k))] \leq q(d(t, t_k)),$$

and

$$\lim_{k \to \infty} r(x(t_k, \omega), x_D(t+, \omega)) = 0,$$

with probability one; i.e. for any $\varepsilon > 0$ and any $\delta > 0$, and for some integer K it holds for every $k > K$

$$P[r(x(t_k), x_D(t+))>\delta/2]\leq\varepsilon.$$

We set, for this δ, $m_\delta=\min\{l;\, g(d(t, t_l))<\delta/2\}$. Then we estimate for every $j>\max(m_\delta, K)$,

$$P[r(x(t), x(t_j))>\delta/2]\leqq q(d(t, t_j)).$$

Hence

$$\begin{aligned}&P[r(x(t), x_D(t+))>\delta]\\ &\leq P[r(x(t), x(t_j))>\delta/2]+P[r(x(t_j), x_D(t+))>\delta/2]\\ &\leq q(d(t, t_j))+\varepsilon.\end{aligned}$$

Thus we get

$$P[r(x(t), x_D(t+))>\delta]=0,$$
$$P[V_t]=1.$$

Clearly, $V_t\cap W_t=\emptyset$, and therefore $P[W_t]=0$. For every t in S and ω, we define

$$y(t, \omega)=\begin{cases}x_D(t+, \omega)\,; & \omega\in V_t,\\ \alpha\in M\,; & \omega\in W_t.\end{cases}$$

It is observed that the stochastic processes $\{y(t, \omega)\}$ is equivalent to the $\{x(t, \omega)\}$ and $y(t, \omega)$ is continuous in t with probability one. The proof of Theorem 1 is completed.

4. Proof of Theorem 2. At first we remark that the condition (3) implies (2). In fact, it is obvious due to the following;

$$\sum_{k=1}^{\infty} g(2^{-k})=\sum_{k=1}^{n} g(2^{-k})+\sum_{k=n+1}^{\infty} g(2^{-k})\leqq\sum_{k=1}^{n} g(2^{-k})+C\cdot g(2^{-n}).$$

This implies the sample-continuity. Next we estimate

$$\begin{aligned}&P\Big[\max_{\substack{t_n\in A_n,\, t_{n+1}\in A_{n+1}\\ d(t_n, t_{n+1})\leqq 2^{-n+2}}} r(x(t_n), x(t_{n+1}))>g(d(t_n, t_{n+1}))\Big]\\ &\leqq\sum_{\substack{t_n\in A_n,\, t_{n+1}\in A_{n+1}\\ d(t_n, t_{n+1})\leq 2^{-n+2}}} P[r(x(t_n), x(t_{n+1}))>g(d(t_n, t_{n+1}))]\\ &\leqq N^2(2^{-n-1})\cdot q(2^{-n+2}).\end{aligned}$$

By (3) and Borel-Cantelli lemma, there exists a number $\nu(\omega)$ with probability one such that, for any $n>\nu(\omega)$ and for any pair (t_n, t_{n+1}), $d(t_n, t_{n+1})\leq 2^{-n+2}$, it holds

$$r(x(t_n), x(t_{n+1}))\leq g(d(t_n, t_{n+1}))\leq g(2^{-n+2}).$$

We shall prove that for a $t_m\in A_m$ satisfying $d(t_m, t_n)\leq 2^{-n+2}$, $t_n\in A_n$, and $m>n>\nu(\omega)$, it holds

$$r(x(t_m), x(t_n))\leq C''\cdot g(2^{-n+1}), \tag{5}$$

where C'' is some positive constant. For such t_m, we can find a sequence $\{t_l\,;\, t_l\in A_l, d(t_m, t_l)\leq 2^{-l}\,;\, l=n, n+1, n+2, \cdots, m\}$. Therefore we get the following estimate; Since $d(t_l, t_{l+1})\leq 2^{-l+1}$,

$$\begin{aligned}r(x(t_m), x(t_n))&\leqq\sum_{k=0}^{m-n-1} r(x(t_{n+k+1}), x(t_{n+k}))\leqq\sum_{k=0}^{\infty} g(2^{-n-k+1})\\ &\leqq g(2^{-n+1})+g(2^{-n})+C\cdot g(2^{-n})\leqq C''\cdot g(2^{-n+1}).\end{aligned}$$

Thus (5) is verified. Further (5) holds for any t in S satisfying $d(t, t_n)\leqq 2^{-n+1}$, $t_n\in A_n$. In fact, taking a sequence $\{t_p\,;\, t_p\in A_p, d(t, t_p)\leq 2^{-p}\,;$

$p=1,2,\cdots\}$, we see that, for any $\varepsilon>0, r(x(t_p), x(t))<\varepsilon$ for sufficiently large p, and that, since for $p>n>\nu(\omega), d(t_p, t_n)\leq d(t_p, t)+d(t, t_n)\leq 2^{-p}+2^{-n+1}\leq 2^{-n+2}$, it holds for sufficiently large p

$$\begin{aligned} r(x(t), x(t_n)) &\leqq r(x(t), x(t_p))+r(x(t_p), x(t_n)) \\ &\leqq \varepsilon + C''\cdot g(2^{-n+1}). \end{aligned}$$

This implies that (5) holds even for $t\in S$ satisfying $d(t, t_n)\leq 2^{-n+1}$, and $n>\nu(\omega)$. Using this fact, we shall show g-Hölder continuity. Set a number $\delta(\omega)=2^{-\nu(\omega)}$. For any pair of points (t, s) such that $d(t, s)<\delta(\omega)$, there exists an integer n satisfying

$$n>\nu(\omega), \quad \text{and} \quad 2^{-n-1}\leq d(t, s)<2^{-n}.$$

On the other hand, we can find a $t_n\in A_n$ satisfying $d(t, t_n)\leq 2^{-n}$. Since $d(t_n, s)\leq d(t_n, t)+d(t, s)\leq 2^{-n}+2^{-n}=2^{-n+1}$, we have

$$\begin{aligned} r(x(t), x(s)) &\leq r(x(t), x(t_n))+r(x(t_n), x(s)) \\ &\leq C''\cdot g(2^{-n+1})+C''g(2^{-n+1}) \\ &\leq 2C''g(4\cdot 2^{-n-1}) \\ &\leq 2C''g(4d(t, s)). \end{aligned}$$

Thus we obtain with probability one, for $d(t, s)<\delta(\omega)$, due to (4)

$$r(x(t), x(s))\leqq C'g(d(t, s)),$$

where $C'=2C''$. This proves Theorem 2.

References

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