174. *Structure of Maximal Sum-free Sets in Groups of Order 3p*

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1. Introduction. In [5] and [6], we studied the structure of maximal sum-free sets of elements in groups of prime orders $p=3k+2$ and $p=3k+1$ respectively. In this paper, we shall study the structure of maximal sum-free sets in groups G (both abelian and non-abelian) of order $3p$, where $p=3k+1$ is a prime. We shall use the same terminologies and notations as used in [1]. In particular, we let S be a maximal sum-free set in G and $|S|$ be the cardinal of S.

2. Abelian groups. Throughout this section G is abelian. We first prove that $|S+S|\neq 2|S|$ in Theorem 4 of [1]. In fact, we shall prove

Lemma 1. *If S is a maximal sum-free set in G, then S is a union of cosets of some subgroup H, of order p or 1, such that*

$$|S+S|=2|S|-|H|.$$

Proof. Write $G=\{0,1,2,\cdots,3p-1\}$. Let $H_0=H=\{0,3,6,\cdots,3(p-1)\}$, $H_1=p+H$, $H_2=2p+H$, $S_i=S\cap H_i$, $i=0,1,2$.

If $S=H_1$, say, then it is clear that $|S+S|\neq 2|S|$.

Assume now that $S\neq H_1$ and $S_1\neq\varnothing$. By Theorem 5 of [1], $|S_0|\leqq k$. Thus $|S_1|+|S_2|\geqq 2k+1$ and without loss of generality, we may assume that $|S_1|\geqq k+1$.

Now $(S_1+S_1)\cap S_2=\varnothing$ and $(S_1+S_1)\cup S_2\subseteqq H_2$. Hence, by Cauchy-Davenport theorem ([2], p. 3), if $S_1+S_1\neq H_2$,

$$\begin{aligned} p &\geqq |S_2|+|S_1+S_1|\geqq |S_2|+2|S_1|-1 \\ &\geqq k+|S_1|+|S_2|\geqq |S_0|+|S_1|+|S_2|=p, \end{aligned}$$

from which it follows that

$$|S_0|=k,\quad |S_1|=k+1,\quad \text{and}\quad |S_2|=k.$$

(If $S_1+S_1=H_2$, then we can prove that $S_0=\varnothing$ and so $S=H_1$, which contradicts the assumption.)

Let $S^*=-S\cup S$. Then $S^*\neq S$. But from Theorem 4 of [1], we have (i) $|S+S|=2|S|-1$ or (ii) $|S+S|=2|S|$ and $S\cup(S+S)=G$. Thus from $S^*\cap(S-S)=\varnothing$ it follows that $|S+S|\neq 2|S|$.

Hence, in any case $|S+S|\neq 2|S|$.

The proof of Lemma 1 is complete.

Next, we prove

Theorem 1. *Let S be a maximal sum-free set in G such that S is* *not a coset of* $H, H=\{0, 3, 6, \cdots, 3(p-1)\}$, *then* S *is given by* $S=S_0 \cup S_1 \cup S_2$, *where*

$$S_0=\{id\,;\ i=k+1, k+2, \cdots, 2k\},$$
$$S_1=p+\{id\,;\ i=0, 1, \cdots, k\},$$
$$S_2=2p+\{id\,;\ i=2k+1, 2k+2, \cdots, 3k\},\ d \in H.$$

Hence the number of maximal sum-free sets S *in* G *such that* S *is not a coset of* H *is* $p-1$. *Moreover, if* S *and* S' *are two maximal sum-free sets in* G *such that* S *and* S' *are not cosets of* H, *then there exists an automorphism* θ *of* G *such that* $S'=S\theta$.

Proof. From the proof of Lemma 1 above, we know that if $S \neq H_1$ and $S_1 \neq \varnothing$ then $|S_0|=k$, $|S_1|=k+1$, $|S_2|=k$, and $|S_1+S_1| = 2|S_1|-1$. Hence by Vosper's theorem ([2], p. 3), S_1 is in arithmetic progression. Let

$$S_1=p+\{a+id\,;\ i=0, 1, \cdots, k\},\ \text{a}, d \in H. \tag{1}$$

Then $S_1-S_1=\{id\,;\ i=0, \pm 1, \cdots, \pm k\}$ and from the fact that $S_0 \cap (S_1-S_1) = \varnothing$ and $|S_0|=k$ it follows that

$$S_0=\{id\,;\ i=k+1, k+2, \cdots, 2k\}. \tag{2}$$

Now, $S_1+S_1=2p+\{2a+jd\,;\ j=0, 1, \cdots, 2k\}$ and from the fact that $S_2 \cap (S_1+S_1)=\varnothing$ and $|S_2|=k$ it follows that

$$S_2=2p+\{2a+id\,;\ i=2k+1, 2k+2, \cdots, 3k\}. \tag{3}$$

Next,

$$S_1+S_2=\{3a+jd\,;\ j=0, 1, \cdots, k-1, 2k+1, 2k+2, \cdots, 3k\}$$

and $S_0 \subseteq H_0 \backslash (S_1+S_2)$, the set complement of S_1+S_2 with respect to H_0. Hence

$$S_0 \subseteq \{3a+id\,;\ i=k, k+1, \cdots, 2k\}. \tag{4}$$

Now by the following lemmas,

Lemma 2. *Let* $A=\{a+jd\,;\ j=0, 1, \cdots, r\}$ *be a set of residues modulo* m *with* $(d, m)=1$ *and* $1 \leqq r \leqq m-3$. *If* $A=\{b+jd'\,;\ j=0, 1, \cdots, r\}$, *then* $d' \equiv \pm d \pmod m$ ([3]).

Lemma 3. *Let* $A=\{a+jd\,;\ j=1, 2, \cdots, r\}$ *be a set of residues modulo* m *with* $(d, m)=1$ *and* $2 \leqq r \leqq (m+1)/2$. *Then* A *can be written in only two essentially different ways in arithmetic progression form, namely*

either $A = \{a+jd\,;\ j=1, 2, \cdots, r\}$

or $A = \{(a+(r+1)d)+j(-d)\,;\ j=1, 2, \cdots, r\}$ ([6]),

we have either $S_0=\{3a+id\,;\ i=k+1, k+2, \cdots, 2k\}$, or $S_0=\{3a+id\,;\ i=k, k+1, \cdots, 2k-1\}$.

Case (i). $S_0=\{3a+id\,;\ i=k+1, k+2, \cdots, 2k\}$. (5)

In this case, compare (5) with (2), we have $a=0$ and thus

$$S_0=\{id\,;\ i=k+1, k+2, \cdots, 2k\}, \tag{2}$$
$$S_1=p+\{id\,;\ i=0, 1, \cdots, k\}, \tag{6}$$
$$S_2=2p+\{id\,;\ i=2k+1, 2k+2, \cdots, 3k\}. \tag{7}$$

Case (ii). $S_0=\{3a+id\,;\,i=k,k+1,\cdots,2k-1\}$. (8)

In this case compare (8) with (2), we have $d=3a$ and therefore $a=-kd$. Thus

$$S_0=\{id\,;\,i=k+1,k+2,\cdots,2k\}, \tag{2}$$
$$S_1=p+\{id\,;\,i=0,2k+1,2k+2,\cdots,3k\}, \tag{9}$$
$$S_2=2p+\{id\,;\,i=1,2,\cdots,k\}. \tag{10}$$

On the other hand, we can verify that $S=S_0\cup S_1\cup S_2$, where S_0, S_1, S_2 are given by (2), (6), and (7) (or (2), (9), and (10)) is sum-free in G and hence is a maximal sum-free set in G.

Now, let

$$S_0'=\{id_0\,;\,i=k+1,k+2,\cdots,2k\}, \tag{2$'$}$$
$$S_1'=p+\{id_0\,;\,i=0,1,\cdots,k\}, \tag{6$'$}$$
$$S_2'=2p+\{id_0\,;\,i=2k+1,2k+2,\cdots,3k\}. \tag{7$'$}$$

We can show that the mapping θ defined by

$$(id)\theta=id_0,\quad (p+id)\theta=p+id_0,$$
$$(2p+id)\theta=2p+id_0,\quad i=0,1,\cdots,p-1$$

is an automorphism of G such that $S\theta=S'$, where $S=S_0\cup S_1\cup S_2$, S_0, S_1, S_2 are given by (2), (6), (7), and $S'=S_0'\cup S_1'\cup S_2'$, S_0', S_1', S_2' are given by (2)$'$, (6)$'$, (7)$'$.

It is clear that the mapping φ defined by

$$(id)\varphi=i(-d),\quad (p+id)\varphi=p+i(-d),$$
$$(2p+id)\varphi=2p+i(-d),\quad i=0,1,\cdots,p-1$$

is an automorphism of G that maps the maximal sum-free set given by (2), (6), and (7) onto the maximal sum-free set given by (2), (9), and (10).

Hence, again, by Lemma 2, there are altogether $p-1$ non-essentially different maximal sum-free sets S in G such that S is not a coset of H. Moreover, all these non-essentially different maximal sum-free sets in G can be obtained by automorphisms from S where $S=S_0\cup S_1\cup S_2$ is given as follows:

$$S_0=\{i\,;\,i=k+1,k+2,\cdots,2k\},$$
$$S_1=p+\{i\,;\,i=0,1,\cdots,k\},\qquad\text{and}$$
$$S_2=2p+\{i\,;\,i=2k+1,2k+2,\cdots,3k\}.$$

The proof of Theorem 1 is complete.

3. Non-abelian groups. Theorem 8 of [1] states that if G is a non-abelian group of order $3p$, where $p=3k+1$ is a prime, then $\lambda(G)=p$. In this section, we shall study the structure of maximal sum-free sets S in G for this case. In fact, we shall prove

Theorem 2. *Let G be a non-abelian group of order $3p$, where $p=3k+1$ is a prime. If S is a maximal sum-free set in G, then S is a coset of a subgroup H, of order p, of G.*

Proof. We know that G is generated by a and b such that $3a=0=pb$ and $b+a=a+rb$, where $r^2+r+1\equiv 0 \pmod p$ ([4], p. 51). It is known that in this case

$$H_0=\{0, b, 2b, \cdots, (p-1)b\}$$

is the only subgroup, of order p, of G ([4], p. 49).

From the proof of Theorem 8 in [1], if S is not a coset of H_0, we can prove that $|S_0|=k$, $|S_1|=k+1$, $|S_2|=k$, and $|S_1+S_1|=2|S_1|-1$ ([1]). Hence, by Vosper's theorem, S_1 is in arithmetic progression. Let

$$S_1=a+\{m+id\,;\, i=0,1,2,\cdots,k\}b$$

where $m, d \in \{0,1,2,\cdots,p-1\}$.

Now $S_1+S_1=2a+\{mr+i(dr)\,;\, i=0,1,2,\cdots,k\}b$
$+\{m+id\,;\, i=0,1,2,\cdots,k\}b$

where $A=\{mr+i(dr)\,;\, i=0,1,2,\cdots,k\}$ and $B=\{m+id\,;\, i=0,1,2,\cdots,k\}$ are elements in the cyclic group C_p of order p.

Again, by Vosper's theorem, A and B should have the same difference. Hence, from Lemma 2, we have $dr\equiv \pm d \pmod p$. But since $d\neq 0$, therefore $r\equiv \pm 1 \pmod p$, which contradicts the fact $r^2+r+1\equiv 0 \pmod p$.

The proof of Theorem 2 is complete.

4. A conjecture. For the case that G is abelian of order 9, the second possibility in Theorem 8 of [1] cannot occur also, i.e., if S is a maximal sum-free set in G, then $|S+S|\neq 2|S|$.

Let H_0 be any subgroup, of order 3, of G. Let H_0, H_1, H_2 be distinct cosets of H_0 and $S_i=S\cap H_i$, $i=0,1,2$.

If the second possibility in Theorem 8 of [1] occurs, then $0\in S+S$ and thus $|(-S)\cap S|=2$. Hence, if $S=\{s_0, s_1, s_2\}$, and $S\neq H_1$ or H_2, then $s_0\in S_0, s_1\in S_1$, and $s_2=-s_1\in S_2$.

Now from $S\cup(S+S)=G$, we have

$$2s_0+(s_0+s_1)+(s_0-s_1)+2s_1+(-2s_1)=-s_0,$$

from which it follows that $5s_0=0$, which is impossible.

We make the following

Conjecture: *Let G be a finite abelian group such that $|G|$ has no prime factors* $\equiv 2 \pmod 3$ *and such that $|G|$ has 3 as a factor. If S is a maximal sum-free set in G, then S is a union of cosets of a subgroup H, of order $|G|/3m$, of G, where m is an integer such that $3m$ $|G|$, and $|S+S|=2|S|-|H|$.*

References

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