144. *On the Global Existence of Real Analytic Solutions of Linear Differential Equations. II*

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§0. In our previous note [9] we have presented a global existence theorem of real analytic solutions for a linear differential operator $P(D)$ with constant coefficients assuming condition (1) below.

(1) The principal symbol $P_m(\xi)$ of $P(D)$ is real and of simple characteristics, i.e., $\mathrm{grad}_\xi P_m \neq 0$ whenever $P_m(\xi)=0$, where ξ is a non-zero real cotangent vector.

That is, denoting by $\mathcal{A}(\Omega)$ the space of real analytic functions on $\Omega \subset \boldsymbol{R}^n$, we have obtained a real analytic solution $u(x)$ of $P(D)u(x)=f(x)$ for any $f(x)$ belonging to $\mathcal{A}(\Omega)$ under some geometrical conditions on Ω. (Kawai [9] Theorem 4.)

The purpose of this note is to extend the results of Kawai [9] in two ways, i.e., in §1 we treat differential operators with constant coefficients not necessarily satisfying condition (1) and in §2 we treat strictly hyperbolic operators with real analytic coefficients defined on a real analytic manifold.

In this note we use the same notations as in our previous note [9] and do not repeat their definitions if there is no fear of confusions.

The details and complete arguments will be given somewhere else.

§1. In this paragraph we use the notion of "localization of differential operators with constant coefficients", which is due to Atiyah, Bott and Gårding [2]. Using the notion of localization Andersson [1] introduced the notion of locally hyperbolic operators and investigated the (analytic) singular support of their elementary solutions. (Andersson [1] Definition 3.2. Such operators are considered also in Kawai [11] independently.) In the sequel we follow Andersson [1] and Atiyah, Bott and Gårding [2] in terminology and notations and do not repeat the definitions: roughly speaking a locally hyperbolic operator with constant coefficients is a differential operator whose localization $P_{\xi_0}(D)$ is hyperbolic with respect to some direction $v(\xi_0)$. The inner core of $P_{\xi_0}(D)$, i.e., the component of $\{\xi \in \boldsymbol{R}^n \mid (P_{\xi_0})_p(\xi)\neq 0\}$ containing $v(\xi_0)$, is denoted by $\Gamma(P_{\xi_0}, v(\xi_0))$ and its dual cone by $K(P_{\xi_0}, v(\xi_0))$, where we denote by $(P_{\xi_0})_p(\xi)$ the principal symbol of $P_{\xi_0}(D)$. We remark that we need not pose any conditions on lower order terms of $P(D)$, which are posed in Andersson [1], since we use hyperfunctions, not distributions, in the below. In the sequel we abbreviate $K(P_{\xi_0}, v(\xi_0))$ to K_{ξ_0} for short.

Theorem 1. *Let a relatively compact domain Ω with smooth boundary in $\boldsymbol{R}^n$ be given in the form $\{x \mid \varphi(x)<0\}$ by a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$. Assume that $P(D)$ is a locally hyperbolic operator and that the domain $\bar{\Omega}$ satisfies the following conditions* (2) *and* (3). *Then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.*

(2) *If x_0 belongs to the boundary of Ω and $P_m(\mathrm{grad}_x\, \varphi(x)|_{x=x_0})=0$, then either $(K_{\xi_0}+x_0)\cap\Omega=\phi$ or $(-K_{\xi_0}+x_0)\cap\Omega=\phi$ holds, where ξ_0 denotes $\mathrm{grad}_x\, \varphi(x)|_{x=x_0}$ and $\pm K_{\xi_0}+x_0$ denotes the translate of the cone $\pm K_{\xi_0}$ whose vertex is at x_0.*

(3) *There exists a family of open sets $\{N_j\}_{j=1}^{l}$ which satisfies the following: For any point x in the boundary of Ω we can find some neighbourhood N_j of x such that for any non-zero real cotangent vector satisfying $P_m(\xi)=0$ either $(K_\xi+x)\cap(\bar{\Omega}\backslash\{x\})\cap N_j=\phi$ or $(-K_\xi+x)\cap(\bar{\Omega}\backslash\{x\})\cap N_j=\phi$ holds.*

The proof of this theorem is just the same as that of Theorem 4 of Kawai [9]. We again emphasize the fact that the theory of sheaf $\mathcal{C}$ (Sato [14]$\sim$[16]), especially the flabbiness of sheaf $\mathcal{C}$ (Kashiwara [5]), plays an essential role in our proof.

§2. In this paragraph we state global existence theorems for strictly hyperbolic operators $P(x, D_x)$ whose coefficients are real analytic functions defined on a real analytic manifold under the assumption of global hyperbolicity on P given in Definition 2 below. Theorem 4 and Theorem 5 treat the global existence of hyperfunction solutions, Theorem 6 and Theorem 7 treat the global existence of real analytic solutions. The main idea of this paragraph which is not included in our previous note [9], i.e., the notion of global hyperbolicity, is due to Leray [12]. (See also the exposition of Bruhat [3].)

In the sequel we consider an m-th order linear differential operator $P(x, D_x)$ with real analytic coefficients on an n-dimensional real analytic manifold V. We assume further that V is endowed with a Riemanian metric and becomes complete with respect to the metric.

Definition 2 (Globally and strictly hyperbolic operators). A linear partial differential operator $P(x, D_x)$ is said to be globally and strictly hyperbolic on V if

(4) It is strictly hyperbolic on V.

and

(5) For any two points x and y in V all time-like paths joining x and y have a bounded length.

This definition is due to Leray [12]. (See also Bruhat [3].) In the sequel we omit the adjective "strict" for short. Using this assumption of global hyperbolicity Leray [12] has given a penetrating study of the geometry of emissions (See Leray [12] Chapter VI § 4 for the definition of emission) and used it to the proof of global existence of solutions in the framework of Sobolev space. (Leray [12] Part II, especially Chapter VII.)

Now we combine his geometrical study of emissions with our construction of local elementary solutions for $P(x, D_x)$ (Kawai [8] § 2, see also Kawai [6]) and our study of the location of its sigularities using the theory of sheaf $\mathcal{C}$ (Kawai [8] § 3, see also Kawai [7]). Then we obtain the following

Theorem 3. *Assume that $P(x, D_x)$ is globally hyperbolic on V in the sense of Definition 2. Then we have the elementary solution $E(x, y)$ for $(x, y) \in V \times V$ satisfying the following conditions (6) and (7).*

(6) *supp $E(x, y) \subset \mathcal{E}(y)$, where $\mathcal{E}(y)$ denotes the emission of y.*

(7) *$S.S.\, E(x, y) \subset \{(x, y; \xi, \eta) \in S^*(V \times V) \mid x=y, \xi=-\eta\} \cup \{(x, y; \xi, \eta) \in S^*(V \times V) \mid (x, \xi)$ and $(y, -\eta)$ are on the same bicharacteristic strip of $P(x, D_x)$ with $x \in \mathcal{E}(y)\}$. Here $S^*(V \times V)$ denotes the cotangential sphere bundle of $V \times V$ and $S.S.\, E(x, y)$ denotes the support of $E(x, y)$ regarded as the section of sheaf $\mathcal{C}$ defined on cotangential sphere bundle, i.e., the support of $\beta(E(x, y))$, where β denotes the surjective sheaf homomorphism from the sheaf of germs of hyperfunctions $\mathcal{B}$ to the direct image of sheaf $\mathcal{C}$ under the canonical projection from the cotangential sphere bundle to the base space. About the details of sheaf $\mathcal{C}$ we refer the reader to Sato* [16].

Using the elementary solution $E(x, y)$ given in Theorem 3 we obviously have the following Theorem 4, since the sheaf of germs of hyperfunctions $\mathcal{B}$ is flabby.

Theorem 4. *Let $P(x, D_x)$ be globally hyperbolic. Then for any relatively compact domain Ω in V we have $P(x, D_x)\mathcal{B}(\Omega)=\mathcal{B}(\Omega)$, where $\mathcal{B}(\Omega)$ denotes the space of hyperfunctions on Ω.*

Remark 1. If $\bar{\Omega}$ is "compact toward the past" (Leray [12] Definition 100), this theorem holds.

Remark 2. Since we consider hyperfunction solutions, we need not pose any conditions on the shape of Ω. (Cf. Harvey [4].)

Theorem 5. *Let $P(x, D_x)$ and Ω be the same as in Theorem 4. Assume further that a non-singular real analytic hypersurface $S=\{x \mid s(x)=0\}$ satisfies the following condition.*

(8) *For any x_0 in S the differential operator $P(x, D_x)$ is hyperbolic with respect to* $\mathrm{grad}_x\, s(x)|_{x=x_0}$.

Then the following Cauchy problem (9) *has a hyperfunction solution* $u(x)$ *if* $f(x)$ *is a hyperfunction defined on* Ω *which depends real analytically on* s.

$$(9)\quad \begin{cases} P(x, D_x)u(x) = f(x) \\ (\partial/\partial s)^j u(x)|_{s=0} = \mu_j(x'), \text{ where } j=0, \cdots, m-1 \text{ and } \mu_j(x') \text{ is a hyperfunction on } S \cap \Omega. \end{cases}$$

Remark 1. If $f(x)$ depends real analytically on s (Sato [13] § 8), Sato's fundamental theorem on the regularity of hyperfunction solutions (Sato [14] ~ [16]) assures the real analytic dependence of $u(x)$ on s. Thus we can consider the restriction of $u(x)$ to S and the above Cauchy problem (9) makes sense.

Remark 2. Let $\tilde{\Omega}$ be the set of all $x \in \Omega$ such that any bicharacteristic curve of P through x intersects $S \cap \Omega$ at some point y and the portion of the bicharacteristic curve with its end points x and y is contained in Ω. Then the solution $u(x)$ of the above Cauchy problem is unique in $\tilde{\Omega}$. This follows trivially from Theorem 3.3 of Kawai [8]. (Cf. Kawai [10], where a precise version of Holmgren's theorem is given. See also Schapira [17] for the usual Holmgren theorem for hyperfunction solutions.)

As for real analytic solutions we have following theorems.

Theorem 6. *Let* $P(x, D_x)$ *be globally hyperbolic. Assume that a relatively compact domain* Ω *in* V *is given in the form* $\{x \mid \varphi(x) < 0\}$ *by a real valued real analytic function* $\varphi(x)$ *defined in a neighbourhood of* $\overline{\Omega}$ *satisfying* $\mathrm{grad}_x\, \varphi \neq 0$ *on* $\partial\Omega$. *If the domain* Ω *satisfies the following conditions* (10) *and* (11), *then* $P(x, D_x)\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$ *holds.*

(10) *If* x_0 *belongs to the boundary of* Ω *and* $P_m(x_0, \mathrm{grad}_x\, \varphi(x)|_{x=x_0}) = 0$, *then the intersection of the bicharacteristic curve through* $(x_0, \mathrm{grad}_x\, \varphi(x)|_{x=x_0})$ *with* Ω *is connected.*

(11) *There exists a family of open sets* $\{N_j\}_{j=1}^{l}$ *which satisfies the following*:
For any point x *in the boundary of* Ω *we can find some* j *such that for any bicharacteristic curve* b *through* (x, ξ) $b \cap (\overline{\Omega} \backslash \{x\}) \cap N_j$ *is connected, where* N_j *is a neighbourhood of* x *and* ξ *is a non-zero real cotangent vector satisfying* $P_m(x, \xi) = 0$.

Theorem 7. *Let* P *and non-singular hypersurface* S *be the same as in Theorem* 5. *Let* I *be a domain in* S *and* Ω *be* $\{x \in V \mid$ *any bicharacteristic curve of* P *through* x *intersects* $I\}$. *Then the following Cauchy problem* (12) *admits a unique real analytic solution in* Ω *for any real analytic function* $f(x)$ *defined on* Ω *and any real analytic function in* $(n-1)$*-variables* $g_j(x')$ *defined on* I.

$$(12)\quad \begin{cases} P(x, D_x)u(x) = f(x) \\ (\partial/\partial s)^j u(x)|_{s=0} = g_j(x'), j=0, \cdots, m-1. \end{cases}$$

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