200. *The Multipliers for Vanishing Algebras*

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Let G be a locally compact Abelian group with Haar measure m. Let Γ be the dual group of G. We denote by $L^1(G)$ the group algebra of G. For any measurable subset S of G, define $L(S)$ to be the subspace of $L^1(G)$ consisting of all functions which vanish locally almost everywhere on the complement of S. When $L(S)$ forms a subalgebra of $L^1(G)$, we call it a vanishing algebra. If $L(S)$ is a vanishing algebra, then we may assume S is a measurable semigroup [2]. In this paper we shall assume $L(S)\neq\{0\}$ to avoid triviality. Let $M(G)$ be the Banach algebra consisting of all bounded regular Borel measures on G. For any Borel set A, put $M(A)=\{\mu\in M(G): \mu \text{ is concentrated on } A\}$.

If A is a Banach algebra, then a mapping $T: A\to A$ is called a multiplier of A if $x(Ty)=(Tx)y (x, y\in A)$.

In this short note, we shall show the characterization of the multipliers for certain vanishing algebras.

Theorem. *If S is an open semigroup, then the space $\mathfrak{M}$ of all multipliers for $L(S)$ is $M(S_0)$, where $S_0=\{t\in G: S\supset S+t$ l.a.e.*)$\}$.*

Proof. At first, we shall show that for any $T\in\mathfrak{M}$ there is a measure $\lambda\in M(G)$ such that $Tf=\lambda*f$ for each $f\in L(S)$ and $\|T\|=\|\lambda\|$. For each $f, g\in L(S)$ we have $(\widehat{Tf})\hat{g}=\hat{f}(\widehat{Tg})$. Since $L(S)$ is contained in no proper colsed ideal of $L^1(G)$ [3], for each $\gamma\in\Gamma$ we can choose a function $g\in L(S)$ such that $\hat{g}(\gamma)\neq 0$. Define $\varphi(\gamma)=(\widehat{Tg})(\gamma)/\hat{g}(\gamma)$. The equation $(\widehat{Tf})\hat{g}=\hat{f}(\widehat{Tg})$ shows that the definition of φ is independent of the choice of g. For φ so defined it is apparent that $(\widehat{Tf})=\varphi\hat{f}$. Let ψ be a second function on Γ such that $(\widehat{Tf})=\psi\hat{f}$ for each $f\in L(S)$. Then since for each $\gamma\in\Gamma$ there is a function $g\in L(S)$ such that $\hat{g}(\gamma)\neq 0$, the equation $(\varphi-\psi)\hat{f}=0$ for each $f\in L(S)$ reveals that $\varphi=\psi$. Evidently, φ is continuous. Let $\gamma_1,\cdots,\gamma_n\in\Gamma$ and $a_1,\cdots,a_n$ be any complex numbers. Let t_0 be a point of S. If $\{x_\alpha\}$ is an approximate identity of $L^1(G)$, then we can assume $(x_\alpha)_{t_0}\in L(S)$, where $(x_\alpha)_{t_0}(t)=x_\alpha(t+t_0)$. Put $b_i=a_i(t_0,\gamma_i)(i=1,2,\cdots,n)$ and $y_\alpha=T((x_\alpha)_{t_0})$. We have that

*) By $A\supset B$ l.a.e., we mean that $B\backslash A$ is locally negligible.

$$\left|\sum_{i=1}^{n} b_i\varphi(\gamma_i)\right| = \left|\sum_{i=1}^{n} b_i \frac{\hat{y}_\alpha(\gamma_i)}{\hat{x}_\alpha(\gamma_i)(t_0, \gamma_i)}\right|$$
$$= \left|\sum_{i=1}^{n} \frac{b_i}{\hat{x}_\alpha(\gamma_i)(t_0, \gamma_i)} \hat{y}_\alpha(\gamma_i)\right|$$
$$= \left|\int_G \left[\sum_{i=1}^{n} \frac{b_i}{\hat{x}_\alpha(\gamma_i)(t_0, \gamma_i)}(-t, \gamma_i)\right] y_\alpha(t) dm(t)\right|$$
$$\leqq \|y_\alpha\| \left\| > \sum_{i=1}^{n} < \frac{b_i}{\hat{x}_\alpha(\gamma_i)(t_0, \gamma_i)}(\cdot, -\gamma_i)\right\|_\infty$$
$$\leqq \|T\| \left\|\sum_{i=1}^{n} \frac{b_i}{\hat{x}_\alpha(\gamma_i)(t_0, \gamma_i)}(\cdot, -\gamma_i)\right\|_\infty .$$

Since $\lim_\alpha x_\alpha(\gamma) = 1$ for each $\gamma \in \Gamma$, we can get

$$\left|\sum_{i=1}^{n} a_i\varphi(\gamma_i)(t_0, \gamma_i)\right| \leqq \|T\| \left\|\sum_{i=1}^{n} a_i(\cdot, -\gamma_i)\right\|_\infty .$$

Appealing now to a well known characterization of Fourier-Stieltjes transforms ([1], p. 32) we conclude there exists a measure $\mu \in M(G)$ such that $\hat{\mu} = (t_0, \cdot)\varphi$ and $\|\mu\| \leqq \|T\|$. Define $\lambda(E) = \mu(E - t_0)$ for any Borel set of G, then $\hat{\lambda} = \varphi$. Thus, $Tf = \lambda * f$ for each $f \in L(S)$. Since $\|Tf\| = \|\lambda * f\| \leqq \|\lambda\| \|f\|$ for each $f \in L(S)$, we have $\|T\| \leqq \|\lambda\|$. It follows that $\|T\| = \|\lambda\|$. Therefore, we may suppose $\mathfrak{M}$ is the closed subalgebra of $M(G)$.

Next, we shall prove that S_0 is a closed semigroup. It is evident that S_0 is a semigroup. Given any $g \in S \backslash S_0$. Since $(S+g)\backslash S$ is non locally negligible, there is a compact subset C of $(S+g)\backslash S$ such that $m(C) > 0$. Let χ_c be a characteristic function of C, then there is a neighborhood V_0 of 0 such that

$$\int_G |\chi_{c+v}(t) - \chi_c(t)|\, dm(t) = m(((C+v)\backslash C) \cup (C\backslash(C+v)))$$
$$< m(C)/2.$$

for any $v \in V_0$ ([1], p. 32). Thus, $m((C+v) \cap C) \geqq m(C)/2 > 0$. Since $(S+g+v)\backslash S \supset (C+v) \cap C$ for each $v \in V_0$, we have that $(V_0+g) \subset G\backslash S_0$. Thus S_0 is closed. Now, we shall show $\mathfrak{M} = M(S_0)$. It is evident $M(S_0) \subset \mathfrak{M}$. Suppose that there is a measure $\mu \in \mathfrak{M}$ such that $\mu \notin M(S_0)$. Then we can assume that μ is a positive measure concentrated on $G\backslash S_0$. Let K be a support of μ. Since $(S+k)\backslash S$ is non locally negligible for any $k \in K \cap (G\backslash S_0)$, there is a non empty compact subset A of $(S+k)\backslash S$ with density property [3]. Put $B = A - k$, then $(B+K)\backslash S$ is non locally negligible. Let V be an open subset of S such that $0 < m(V) < \infty$ and $B \cap V \neq \emptyset$. Since $\{(B \cap V)+K\}\backslash S \supset A \cap (V+k) \neq \emptyset$, $\{(B \cap V)+K\}\backslash S$ is non locally negligible. If $x \in (B \cap V)+K$, then since $(x-V) \cap K \neq \emptyset$, $0 < \mu((x-V) \cap K) < \infty$. Let χ be a characteristic function of V, then $\chi \in L(S)$. We see that

$$\begin{aligned}\chi * \mu(x) &= \int_G \chi(x-y)\,d\mu(y)\\ &= \int_K \chi(x-y)\,d\mu(y)\\ &= \mu((x-V)\cap K) > 0\end{aligned}$$

for each $x \in (V+K)\backslash S$. Since $(V+K)\backslash S$ is non locally negligible, $\chi * \mu \notin L(S)$. This completes the proof.

References

[1] W. Rudin: Fourier Analysis on Groups. Interscience, New York (1962).

[2] T. Shimizu: On a problem of vanishing algebras. Proc. Japan Acad., **46**, 3–4 (1970).

[3] A. B. Simon: Vanishing algebras. Trans. Amer. Math. Soc., **92**, 154–167 (1959).