178. On Countably R-closed Spaces. I

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A topological space S is called *countably R-closed*, if for any family $\{G_n\}_{n=1}^{\infty}$ of nonvoid open sets such that $G_n \supset \bar{G}_{n+1}$ for every n, we have $\bigcap_{n=1}^{\infty} G_n \neq \phi$. Z. Frolik [1] proved the following:

Proposition. *In any topological space S, the following properties are equivalent*:

(i) *S is countably R closed.*

(ii) *Every star-finite open covering of S has a finite subfamily whose union is dense in S.*

(iii) *Every star-finite open covering of S has a finite subcovering.*

(iv) *Every star-finite open covering of S is a finite covering.*

We shall give other characterizations of countably R-closed spaces. In a topological space S, a family Φ composed of subsets of S is called *locally finite* (*discrete*) if every point x has a neighbourhood $U(x)$ which meets only finite members (at most only one member) of Φ, and Φ is called *star-finite* if every member of Φ meets only finite members of Φ. A subset E is called *regularly closed* if E is the closure of an open set of S. A covering of S composed of regularly closed sets is called *a regularly closed covering of S.*

Theorem. *In any topological space S, the following conditions are equivalent*:

(1) *S is countably R-closed.*

(2) *Every locally finite, star-finite, countable, regularly closed covering of S has a finite subcovering.*

(3) *Every locally finite, star-finite, countable, regularly closed covering of S is a finite covering.*

(4) *Every locally finite, star-finite, regularly closed covering of S is a finite covering.*

(5) *Every star-finite open covering of S is a finite covering.*

We shall prove that (1)→(2)→(3)→(1) and (3)→(4)→(5)→(4)→(3).

Lemma 1. *In a topological space S, let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, countable, regularly closed covering of S. Then $F_n = \bigcup_{k=n+1}^{\infty} \bar{O}_k$ is closed, $G_n = S - \bigcup_{l=1}^{n} \bar{O}_l$ is open, and $F_n \supset G_n$ for every n.*

Lemma 2. *Let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, star-finite, countable, regularly closed covering of S. For every n, there is m ($\geqq n+1$) such that $F_m \subset G_n$.*

Proof that (1)→(2). Let S be a topological space and let $\{\bar{O}_n\}_{n=1}^{\infty}$ be a locally finite, star-finite, countable, regularly closed covering of S. Suppose that $\{\bar{O}_n\}_{n=1}^{\infty}$ has no finite subcovering of S. Then $S-\bigcup_{l=1}^{n}\bar{O}_l\neq\phi$ i.e. $G_n\neq\phi$ for every n. Put $n_1=1, F_{n_1}=\bigcup_{k=2}^{\infty}\bar{O}_k$ and $G_{n_1}=S-\bar{O}_1$. By Lemma 1, F_{n_1} is closed, G_{n_1} is open, and $F_{n_1}\supset G_{n_1}$. Suppose that a sequence $\{n_t\}_{t=1}^{s}$ has defined such that $n_1<n_2<\cdots<n_s, F_{n_t}$ $(1\leqq t\leqq s)$ are closed, G_{n_t} $(1\leqq t\leqq s)$ are nonvoid open, and $F_{n_1}\supset G_{n_1}\supset F_{n_2}\supset G_{n_2}\supset\cdots\supset F_{n_s}\supset G_{n_s}$. By Lemma 2, for n_s there is m $(\geqq n_s+1)$ such that $G_{n_s}\supset F_m\supset G_m$ where F_m is closed and G_m is nonvoid open. Put $n_{s+1}=m$. Thus we have a sequence $\{n_t\}_{t=1}^{s+1}$ such that $n_1<n_2<\cdots<n_s<n_{s+1}, F_{n_t}$ $(1\leqq t\leqq s+1)$ are closed, G_{n_t} $(1\leqq t\leqq s+1)$ are nonvoid open, and $F_{n_1}\supset G_{n_1}\supset\cdots\supset F_{n_{s+1}}\supset G_{n_{s+1}}$. Then there is a sequence $\{n_t\}_{t=1}^{\infty}$ such that $n_1<n_2<\cdots<n_t<\cdots$, F_{n_t} $(1\leqq t<\infty)$ are closed, G_{n_t} $(1\leqq t<\infty)$ are nonvoid open, and $F_{n_1}\supset G_{n_1}\supset\cdots\supset F_{n_t}\supset G_{n_t}\supset\cdots$ i.e. $G_{n_1}\supset\bar{G}_{n_2}\supset G_{n_2}\supset\cdots\supset G_{n_t}\supset\bar{G}_{n_{t+1}}\supset\cdots$. If the space S were countably R-closed, we should have $\bigcap_{t=1}^{\infty}G_{n_t}\neq\phi$. Then, there should exist a point x_0 such that $x_0\in G_{n_t}$ i.e. $x_0\bar{\in}\bigcup_{l=1}^{n_t}\bar{O}_l$ for every t. Then $x_0\bar{\in}\bigcup_{l=1}^{\infty}\bar{O}_l$, contrary to that $\{\bar{O}_n\}_{n=1}^{\infty}$ is a covering of S. Therefore S may not be countably R-closed.

Proof that (2)→(3). Let S be a topological space and let us assume that any locally finite, star-finite, countable, regularly closed covering $\{\bar{H}_n\}_{n=1}^{\infty}$ of S, has a finite subcovering $\{\bar{H}_{n_k}\}_{k=1}^{m}$. If infinitely many different $\bar{H}_n$ were not nonvoid, there should exist a member $\bar{H}_{n_k}$ which met infinitely many different nonvoid $\bar{H}_n$. This is contrary to the condition that $\{\bar{H}_n\}_{n=1}^{\infty}$ is star-finite. Then the family $\{\bar{H}_n\}_{n=1}^{\infty}$ must be finite.

Lemma 3. *In any topological space S, if there is a family $\{G_n\}_{n=1}^{\infty}$ of open sets where $G_n\supset\bar{G}_{n+1}$ for every n and $\bigcap_{n=1}^{\infty}G_n=\phi$, the family $\{\bar{H}_n\}_{n=0}^{\infty}$, where $H_0=S-\bar{G}_1$ and $H_n=G_n-\bar{G}_{n+1}$ for every n $(\geqq 1)$, is a locally finite, star-finite, regularly closed covering of S.*

Proof. Put $K_0=S-\bar{G}_3$, $K_n=G_n-\bar{G}_{n+3}$ for every n $(\geqq 1)$. In the first place, we shall prove that $\bar{H}_0\subset K_0, \bar{H}_n\subset K_{n-1}$ for every n $(\geqq 1)$. Since $H_0=S-\bar{G}_1\subset S-G_1$ and $S-G_1$ is closed, we have $\bar{H}_0\subset S-G_1\subset S-\bar{G}_3=K_0$. Since $H_n=G_n-\bar{G}_{n+1}\subset\bar{G}_n-G_{n+1}$ and $\bar{G}_n-G_{n+1}$ is closed for every n $(\geqq 1)$, we have $\bar{H}_n\subset\bar{G}_n-G_{n+1}\subset G_{n-1}-\bar{G}_{n+2}=K_{n-1}$, assuming that $G_0=S$. Since the family $\{K_n\}_{n=0}^{\infty}$ of open sets is star-finite covering of S and $\bar{H}_0\subset K_0$, $\bar{H}_n\subset K_{n-1}$ for every n $(\geqq 1)$, we have that the family $\{\bar{H}_n\}_{n=0}^{\infty}$ is locally finite and star-finite. Nextly we shall prove that $\{\bar{H}_n\}_{n=0}^{\infty}$ is a covering of the space S. Since the family $\{K_n\}_{n=0}^{\infty}$ is a covering of S and $K_n=G_n-\bar{G}_{n+3}=H_n\cup(\bar{G}_{n+1}-G_{n+1})\cup H_{n+1}\cup(\bar{G}_{n+2}-G_{n+2})\cup H_{n+2}$ for every n $(\geqq 0)$, it is sufficient to prove that $\bar{G}_n-G_n\subset\bar{H}_n$ for every n $(\geqq 1)$. If $\bar{G}_n-G_n=\phi$, it is trivial. If $\bar{G}_n-G_n\neq\phi$, let x be an arbitrary point of $\bar{G}_n-G_n$. Since $x\in\bar{G}_n-G_n\subset G_{n-1}-\bar{G}_{n+1}$ and $G_{n-1}-\bar{G}_{n+1}$ is open, $G_{n-1}-\bar{G}_{n+1}$ is a neighbourhood $V_1(x)$ of x. Since $x\in\bar{G}_n-G_n\subset\bar{G}_n$, for any neighbourhood $V(x)$ of x, we have $V(x)\cap V_1(x)\cap G_n\neq\phi$. Hence, for any neighbourhood $V(x)$ of x,

$$V(x)\cap H_n=V(x)\cap(G_n-\bar{G}_{n+1})=V(x)\cap(G_{n-1}-\bar{G}_{n+1})\cap G_n$$
$$=V(x)\cap V_1(x)\cap G_n\neq\phi.$$

Then $x\in\bar{H}_n$ i.e. $\bar{G}_n-G_n\subset\bar{H}_n$. Thus we have proved that $\{\bar{H}_n\}_{n=0}^{\infty}$ is a covering of the space S.

Proof that (3)→(1). Let us assume that in a topological space S any locally finite, star-finite, countable, regularly closed covering $\{\bar{O}_n\}_{n=1}^{\infty}$ of S is a finite covering. If there is a family $\{G_n\}_{n=1}^{\infty}$ of open sets where $G_n\supset\bar{G}_{n+1}$ for every n and $\bigcap_{n=1}^{\infty}G_n=\phi$, we put $H_0=S-\bar{G}_1$, $H_n=G_n-\bar{G}_{n+1}$ for every n ($\geqq 1$). By Lemma 3, the family $\{\bar{H}_n\}_{n=0}^{\infty}$ is a locally finite, star-finite countable, regularly closed covering of S. Then, by the above assumption of the space S, the covering $\{\bar{H}_n\}_{n=0}^{\infty}$ is a finite covering. Hence, there is N such that for every $n\geqq N$, $\bar{H}_n=\phi$ i.e. $H_n=\phi$ from which we have $G_n=\bar{G}_{n+1}$. In the proof of Lemma 3, $\bar{G}_n-G_n\subset\bar{H}_n$ for every n ($\geqq 1$), then we have $\bar{G}_n-G_n=\phi$ i.e. $\bar{G}_n=G_n$ for every $n\geqq N$. Therefore, we have $\bar{G}_N=G_N=\bar{G}_{N+1}=G_{N+1}=\cdots$. Then we have $\bigcap_{n=1}^{\infty}G_n=G_N$. From $\bigcap_{n=1}^{\infty}G_n=\phi$, we have $G_N=\phi$. Then for any family $\{G_n\}_{n=1}^{\infty}$ of nonvoid open sets such that $G_n\supset\bar{G}_{n+1}$ for every n, we must have $\bigcap_{n=1}^{\infty}G_n\neq\phi$, that is, the space S must be countably R-closed.

Lemma 4. *Let $\{\bar{O}_\alpha\}_{\alpha\in A}$ be a locally finite, star-finite, regularly closed covering of any topological space S. We have the following properties.*

(i) *The family $\{\bar{O}_\alpha\}_{\alpha\in A}$ is decomposed into components $\{K_\gamma\}_{\gamma\in\Gamma}$, where $K_\gamma=\{\bar{O}_\alpha\}_{\alpha\in A_\gamma}$ and $K_\gamma(\gamma\in\Gamma)$ are pairwise disjoint.*

(ii) *The family $K_\gamma=\{\bar{O}_\alpha\}_{\alpha\in A_\gamma}$ is a countable family for every γ.*

(iii) *The union $R_\gamma=\bigcup_{\alpha\in A_\gamma}\bar{O}_\alpha$ is open and closed for every γ.*

(iv) *The family $\{R_\gamma\}_{\gamma\in\Gamma}$ of open and closed sets is a discrete covering of S.*

Proof of (i). Let $\bar{O}_\alpha$ be an arbitrary member of $\{\bar{O}_\alpha\}_{\alpha\in A}$. Let us define $S^n(\bar{O}_\alpha)$ for $n=1,2,\cdots$ as follows:

$$S^1(\bar{O}_\alpha)=\bigcup_\beta\{\bar{O}_\beta;\ \bar{O}_\beta\cap\bar{O}_\alpha\neq\phi,\ \beta\in A\},$$
$$S^{n+1}(\bar{O}_\alpha)=\bigcup_\beta\{\bar{O}_\beta;\ \bar{O}_\beta\cap S^n(\bar{O}_\alpha)\neq\phi,\ \beta\in A\}\qquad\text{for every } n.$$

For any $\bar{O}_\alpha$ and $\bar{O}_\beta$, if there is n such that $\bar{O}_\beta\subset S^n(\bar{O}_\alpha)$, we denote $\bar{O}_\beta\sim\bar{O}_\alpha$, which is an equivalence relation. Therefore the family $\{\bar{O}_\beta\}_{\beta\in A}$ is decomposed into components $\{K_\gamma\}_{\gamma\in\Gamma}$ where $K_\gamma=\{\bar{O}_\beta\}_{\beta\in A_\gamma}$ and $K_\gamma\cap K_{\gamma'}=\phi$ for any γ and γ' ($\gamma\neq\gamma'$).

Lemma 5. *In any countably R-closed space S, if a subset E is open and closed, then the subspace E is countably R-closed.* (See Z. Frolik [1], p. 217.)

Lemma 6. *In a countably R-closed space S, any discrete open covering is a finite covering.*

Proof that (3)→(4). Let S be a topological space having the property (3). By the implication (3)→(1), which has been shown, S is countably R-closed. Let $\{\bar{O}_\alpha\}_{\alpha\in A}$ be a locally finite, star-finite, regularly closed covering of S. By Lemma 4, the family $\{R_\gamma\}_{\gamma\in\Gamma}$ is a discrete covering of S composed of open and closed sets. By Lemma 6, the family $\{R_\gamma\}_{\gamma\in\Gamma}$ is a finite covering of S. Since the set R_γ is open and closed for every γ, by Lemma 5, the subspace R_γ is countably R-closed space. By Lemma 4, for every $\gamma\in\Gamma$, the family $K_\gamma=\{\bar{O}_\alpha\}_{\alpha\in A_\gamma}$ is a locally finite, star-finite, countable, regularly closed covering of the countably R-closed space R_γ. By the implications (1)→(2)→(3), which have been shown, the family $K_\gamma=\{\bar{O}_\alpha\}_{\alpha\in A_\gamma}$ is a finite covering of R_γ. Therefore the covering $\{\bar{O}_\alpha\}_{\alpha\in A}$ is a finite covering of S.

Proof that (4)→(3). It is obvious.

Proof that (4)→(5). From the proposition and the implications (1)⇄(3)⇄(4), the implications (4)⇄(5) are obvious. We shall prove directly that (4)→(5). Let S be a topological space which has the property (4) and let $\{O_\alpha\}_{\alpha\in A}$ be a star-finite open covering of S. It is obvious that the family $\{\bar{O}_\alpha\}_{\alpha\in A}$ is a locally finite, star-finite, regularly closed covering of S, from which, by the property (4), the family $\{\bar{O}_\alpha\}_{\alpha\in A}$ is finite i.e. the family $\{O_\alpha\}_{\alpha\in A}$ is finite, since $\bar{O}_\alpha\subset\bigcup_\beta\{O_\beta\,;\,O_\beta\cap O_\alpha\neq\phi,\ \beta\in A\}$ for every $\alpha\in A$.

Lemma 7. *Let $\{\bar{O}_\alpha\}_{\alpha\in A}$ be a locally finite, star-finite, regularly closed covering of any topological space S and let $S(\bar{O}_\alpha)=\bigcup_\beta\{\bar{O}_\beta;\bar{O}_\beta\cap\bar{O}_\alpha\neq\phi,\ \beta\in A\}$ for every $\bar{O}_\alpha$, then we have $\bar{O}_\alpha\subset\operatorname{Int}S(\bar{O}_\alpha)$.*

Proof that (5)→(4). We shall prove directly that (5)→(4). Let S be a topological space which satisfies the condition (5) and let $\{\bar{O}_\alpha\}_{\alpha\in A}$ be a locally finite, star-finite, regularly closed covering of S. Put $G_\alpha=\operatorname{Int}S(\bar{O}_\alpha)$ for every $\alpha\in A$. It is easily proved that the family $\{S(\bar{O}_\alpha)\}_{\alpha\in A}$ is a locally finite, star-finite, regularly closed covering of the space S. It is also easily proved that the family $\{G_\alpha\}_{\alpha\in A}$ is star-finite. By Lemma 7, $G_\alpha\supset\bar{O}_\alpha$ for every α, then $\{G_\alpha\}_{\alpha\in A}$ is a star-finite open covering of S. By the condition (5), the covering $\{G_\alpha\}_{\alpha\in A}$ is finite. Since $S(\bar{O}_\alpha)\supset G_\alpha$, $\{S(\bar{O}_\alpha)\}_{\alpha\in A}$ is a finite covering. Being star-finite, $\{\bar{O}_\alpha\}_{\alpha\in A}$ is a finite covering.

References

[1] Z. Frolík: Generalisations of compact and Lindelöf spaces. Czecho. Math. Jour., **9**(84), 172–217 (1958).

[2] K. Iséki: Generalisation of the notion of compactness. Rev. Math. Pures Appl., **6**, 31–63 (1961).