230. On the Radon Transform of the Rapidly Decreasing Functions on Symmetric Spaces

By Masaaki EGUCHI
Faculty of General Education, Hiroshima University

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1. Let S be a Riemannian globally symmetric space and $\hat{S}$ the Radon dual space of S consisting of the holocycles in S. The purpose of this paper is to study the relations between the Schwartz functions on S and those on $\hat{S}$, that is, to study an $\mathcal{S}$-theory in a sense. For the detailed proof, see [1].

2. The Schwartz spaces. Let G denote the largest connected group of isometries of S in compact open topology. Let o be any point in S, K the isotropy subgroup of G at o and $\mathfrak{k}_0$ and $\mathfrak{g}_0$ their Lie algebras, respectively. Let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be the corresponding Cartan decomposition of $\mathfrak{g}_0$. Let $\mathfrak{h}_{\mathfrak{p}_0}$ denote a Cartan subalgebra for the space S, $A_{\mathfrak{p}}$ the analytic subgroup of G corresponding to $\mathfrak{h}_{\mathfrak{p}_0}$ and M the centralizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in K. Let extend $\mathfrak{h}_{\mathfrak{p}_0}$ to a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, of the corresponding roots let P_+ denote the set of those whose restriction to $\mathfrak{h}_{\mathfrak{p}_0}$ is positive in the ordering defined by a fixed Weyl chamber C in $\mathfrak{h}_{\mathfrak{p}_0}$. Then we obtain an Iwasawa decomposition $G=KA_{\mathfrak{p}}N$. Put $\rho=\frac{1}{2}\sum_{\alpha\in P_+}\alpha$ as usual. Let $\boldsymbol{D}(S)$(resp. $\boldsymbol{D}(\hat{S})$) denote the algebra of G-invariant differential operators on S (resp. $\hat{S}$) and $\hat{\boldsymbol{D}}$ the image of the isomorphism of $\boldsymbol{D}(S)$ into $\boldsymbol{D}(\hat{S})$.

For $x\in S=G/K$ and $g\in G$ such that $\pi(g)=x$ by the natural mapping π of G onto G/K, there exists a unique element $X\in\mathfrak{p}_0$ such that $x=\exp X\cdot K$. Now put

$$\omega(x)=\{\det(\sinh \operatorname{ad} X/\operatorname{ad} X)\}^{1/2},$$

$$\sigma(g)=\sigma(x)=\|X\|,$$

$$\xi(x)=\int_K \exp\{-\rho(H(\exp X\cdot k))\}dk.$$

For $f\in C^\infty(S)$, $D\in\boldsymbol{D}(S)$ and integer $d\geqq 0$, put

$$\nu_{D,d}(f)=\sup_S |Df|(1+\sigma)^d\xi^{-1},$$

$$\tau_{D,d}(f)=\sup_S |Df|(1+\sigma)^d\omega.$$

We now define the Schwartz space after Harish-Chandra [2].

Definition 1. *Let $\mathcal{C}(S)$(resp. $\mathcal{S}(S)$) denote the space of all $f\in C^\infty(S)$ such that $\nu_{D,d}(f)<+\infty$ (resp. $\tau_{D,d}(f)<+\infty$) for all $D\in\boldsymbol{D}(S)$ and integers $d\geqq 0$.*

We topologize $\mathcal{C}(S)$ (resp. $\mathcal{S}(S)$) by means of the system of semi-norms $\nu_{D,d}$ (resp. $\tau_{D,d}$) ($D \in \boldsymbol{D}(S)$, $d \geqq 0$). And we call $\mathcal{C}(S)$ the Schwartz space of S. Let ψ denote the diffeomorphism $(kM, h) \mapsto khMN$ of $K/M \times A_{\mathfrak{p}}$ onto $\hat{S}$.

Definition 2. *Let $\mathcal{S}(\hat{S})$ denote the set of all functions $\varphi \in C^\infty(\hat{S})$ which satisfy the following conditions: For all $\hat{D} \in \hat{\boldsymbol{D}}$ and integers $r \geqq 0$ and real numbers t,*

$$\mu_{\hat{D},r,t}(\varphi) = \sup_{\substack{h \in A_{\mathfrak{p}} \\ \rho(\log h) \geqq t}} (1 + \|\log h\|)^r \, |[(\hat{D}\varphi) \circ \psi](kM, h)| < +\infty.$$

3. **The theorems.** Let $\hat{f}$ denote the Radon transform of the function f on S, that is, for a normalized measure dn on N

$$\hat{f}(gMN) = \int_N f(gn \cdot o) dn.$$

Let $\check{\varphi}$ denote the inverse transform of the continuous function φ on $\hat{S}$, that is,

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gkMN) dk.$$

Then we obtain the following theorems.

Theorem A. *For any $f \in \mathcal{C}(S)$ and $D \in \boldsymbol{D}(S)$*

$$\widehat{Df} = \hat{D}\hat{f}.$$

Theorem B. *The mapping $f \mapsto \hat{f}$ is a one-to-one continuous linear mapping of $\mathcal{S}(S)$ into $\mathcal{S}(\hat{S})$.*

As a corollary of this theorem, we obtain the following

Theorem B′. *The mapping $f \mapsto \hat{f}$ is a one-to-one continuous linear mapping of $\mathcal{C}(S)$ into $\mathcal{S}(\hat{S})$.*

Theorem C. *If G has a complex structure then there exists an explicit differential operator $\Box \in \boldsymbol{D}(S)$ such that*

$$\Box((\hat{f})^\vee) = f, \qquad \text{for any } f \in \mathcal{C}(S).$$

Theorem D. *Let $\check{E} \in \boldsymbol{D}(S)$ correspond to $E \in \hat{\boldsymbol{D}}$ under theiso morphism $\boldsymbol{D}(S) \cong \hat{\boldsymbol{D}}$. For any function φ in the image of $\mathcal{C}(S)$, by the Radon transform, the following relation holds*

$$(E\varphi)^\vee = \check{E}\check{\varphi}.$$

References

[1] M. Eguchi: On the Radon transform of the rapidly decreasing functions on symmetric spaces. II. Hiroshima Math. J., **1**, 161–169 (1971).

[2] Harish-Chandra: Discrete series for semisimple Lie groups. II. Acta Math., **116**, 1–111 (1966).