219. On Ergodic and Abelian Automorphism Groups of von Neumann Algebras

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Recently, in [5] Tam proved that any ergodic and abelian automorphism group of an abelian von Neumann algebra is freely acting.

In this paper, we shall give a generalization of Tam's theorem, using the notion of the generalized free action due to Kallman [3]. And we shall generalize Kallman's theorem that all the powers of an ergodic automorphism of a II_1-factor are outer [3].

1. Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathfrak{H}$. In this paper we shall write briefly a *-automorphism of $\mathcal{A}$ as an automorphism of $\mathcal{A}$.

Definition A. Let G be a group of automorphisms of a von Neumann algebra $\mathcal{A}$. Then G is called to be *ergodic* on $\mathcal{A}$ if the only A in $\mathcal{A}$ which satisfies

$$(*) \qquad g(A)=A \qquad (\text{for all } g \in G)$$

is scalar. An automorphism g on $\mathcal{A}$ is called to be *ergodic* on $\mathcal{A}$ if the only A in $\mathcal{A}$ which satisfies the condition (*) is scalar.

Kallman [3] has generalized the von Neumann free action for an abelian von Neumann algebra as follows:

Definition B (Kallman). An automorphism g on a von Neumann algebra $\mathcal{A}$ is called to be *freely acting* on $\mathcal{A}$ if the only A in $\mathcal{A}$ which satisfies

$$(**) \qquad AB=g(B)A \qquad \text{for all } B \in \mathcal{A}$$

is $A=0$.

The condition (**) in the Definition B is used by Nakamura and Takeda and plays an important role in the theory of the crossed product [4].

Under the Definition B, Kallman showed that any automorphism of a von Neumann algebra is decomposed into freely acting part and inner part. Using this theorem, we have the following:

Lemma 1. *Let $\mathcal{A}$ be a von Neumann algebra, G an ergodic group of automorphisms of $\mathcal{A}$ and α an automorphism of $\mathcal{A}$ such that*

$$\alpha g = g\alpha \qquad \textit{for every } g \in G.$$

Then the automorphism α is freely acting or inner.

Proof. By the Kallman theorem, there exist a central projection P and a unitary operator U in $\mathcal{A}$ such that

$$\alpha(AP) = U^*APU \qquad \text{for any } A \text{ in } \mathcal{A},$$

and that α is freely acting on $\mathcal{A}_{I-P}$.

For any element g of G,

$$\alpha(g(P)) = g(\alpha(P)) = g(P)$$

and

$$\begin{aligned}\alpha(g(P)B) &= g(\alpha(Pg^{-1}(B))) \\ &= g(U^*g^{-1}(B)PU) \\ &= g(U)^*Bg(P)g(U) \qquad \text{for any } B \text{ in } \mathcal{A}.\end{aligned}$$

Then, by the definition of P [3, Proof of Theorem 1], we have

$$g(P) \leqq P \qquad \text{for any } g \in G.$$

On the other hand, since G is ergodic,

$$\sup\{g(P)\,;\, g \in G\} = 0 \quad \text{or} \quad I.$$

Therefore $P=0$ or I, that is, α is inner or freely acting.

By Lemma 1, we have the following generalized Tom's theorem:

Theorem 2. *Let $\mathcal{A}$ be a von Neumann algebra and G an ergodic and abelian group of outer automorphisms of $\mathcal{A}$. Then G is freely acting on $\mathcal{A}$.*

Proof. For any element $g \in G$ ($g \neq e$, unit of G), we have

$$gh = hg \qquad \text{for any } h \in G.$$

Then, by Lemma 1, g is freely acting or inner.

Therefore, since G is a group of outer automorphisms of $\mathcal{A}$, g is freely acting on $\mathcal{A}$. That is, G is freely acting.

2. The following lemma may be known in the specialist.

Lemma 3. *Let $\mathcal{A}$ be a continuous von Neumann algebra acting on $\mathfrak{H}$ and $\mathcal{B}$ a maximal abelian subalgebra of $\mathcal{A}$. Then for any nonzero projection P in $\mathcal{B}$ there exist two orthogonal nonzero projections Q and R in $\mathcal{B}$ such that*

$$P = Q + R.$$

Proof. It is sufficient to show that $\mathcal{B}$ does not have any minimal projection in $\mathcal{B}$. If there is a minimal projection P in $\mathcal{B}$, then by the minimality of P, the reduced von Neumann algebra $\mathcal{B}_P$ is the algebra $\mathcal{C}_{P(\mathfrak{H})}$ of scalar multiples of the identity on $P(\mathfrak{H})$. On the other hand, since $\mathcal{B}$ is a maximal abelian subalgebra of $\mathcal{A}$, $\mathcal{B}_P$ is a maximal abelian subalgebra of $\mathcal{A}_P$ [1, p. 13 and p. 18]. Then $\mathcal{B}_P$ equals to $\mathcal{A}_P$ by the following equality;

$$\mathcal{A}_P = \mathcal{A}_P \cap \mathcal{C}_P' = \mathcal{A}_P \cap \mathcal{B}_P' = \mathcal{B}_P.$$

This contradicts that $\mathcal{A}_P$ is continuous [1, p. 125].

It is known that all the powers of an ergodic measure preserving automorphism on a non-atomic probablility measure space are freely acting [2]. As an analogous statement for II_1-factor, Kallman proved in [3] that all the powers of an ergodic automorphism of II_1-factor are outer. We have a generalization of this theorem as follows:

Theorem 4. *Let $\mathcal{A}$ be a continuous von Neumann algebra acting on a Hilbert space $\mathfrak{H}$ and g an ergodic automorphism of $\mathcal{A}$. Then g^n $(n=\pm 1, \pm 2, \cdots)$ is freely acting on $\mathcal{A}$.*

Proof. If g^n does not freely acting for some n $(=\pm 1, \pm 2, \cdots)$, then by Lemma 1 g^n is inner. We may assume that $n>0$. Then there exists a unitary operator U in $\mathcal{A}$ such that

$$g^n(A)=U^*AU \qquad \text{for all } A \text{ in } \mathcal{A}.$$

Let $\mathcal{B}$ be a maximal abelian subalgebra of $\mathcal{A}$ containing U. Then we have

$$g^n(B)=B \qquad \text{for all } B \text{ in } \mathcal{B}.$$

Let, for any nonzero projection Q in $\mathcal{B}$,

$$R=Q+g(Q)+\cdots+g^{n-1}(Q).$$

Then $g(R)=R$, so R is some scalar multiple of the identity, say $R=\lambda I$. Since Q is a nonzero projection, we have $\lambda \geqq 1$.

Take a unit vector x in $\mathcal{H}$. For a natural number k with $k>n^2$, there exist k mutually orthogonal projections Q_i in $\mathcal{B}$ $(i=1, 2, \cdots, k)$ with $\sum_{i=1}^{k} Q_i=I$, by Lemma 3. By the equality;

$$1=\|Ix\|^2=\sum_{i=1}^{k}\|Q_i x\|^2,$$

there exists i such as

$$\|Q_i x\|<1/n.$$

If $\|g(Q_i)x\| \geqq 1/n$, we choose again k mutually orthogonal nonzero projections R_j $(j=1, 2, \cdots, k)$ in $\mathcal{B}$ with $Q_i=\sum_{j=1}^{k} R_j$, by Lemma 3. As such as Q_i,

$$1 \geqq \|g(Q_i)x\|^2=\sum_{j=1}^{k}\|g(R_j)\|^2,$$

then there exists a nonzero projection R_j in $\mathcal{B}$ with $\|g(R_j)x\|<1/n$. Then we have a nonzero projection Q in $\mathcal{B}$ such that

$$\|Qx\|<1/n$$

and

$$\|g(Q)x\|<1/n.$$

Going on this method, we have a nonzero projection Q in $\mathcal{B}$ such as for any k $(1 \leqq k \leqq n)$,

$$\|g^k(Q)x\|<1/n.$$

Since, for this nonzero projection Q in $\mathcal{B}$,

$$\lambda x=Rx=Qx+g(Q)x+\cdots+g^{n-1}(Q)x,$$

$|\lambda|<1$ which contradicts $\lambda \geqq 1$.

By the proof of Theorem 4, we can see that Theorem 4 is valid for a nonatomic abelian von Neumann algebra.

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