213. *On the Structure of Hyperfunctions with Compact Supports*

By Akira KANEKO*)
Department of Mathematics, University of Tokyo

(Comm. by Kunihiko KODAIRA, M. J. A., Sept. 13, 1971)

We discuss an analogue of the classical structure theorem of distributions on a compact set. We mainly treat the case of one variable ($n=1$). The case of several variables with some applications will be discussed by a somewhat different method in a paper under preparation (see [3]).

Theorem 1. *Let u be a hyperfunction of one variable with support in the interval $K=[a,b]$. Then u can be expressed as follows:*

$$u=J_1(D)\mu_1+J_2(D)\mu_2+J_3(D)\mu_3,$$

where μ_i, $i=1,2,3$ are measures with supports in $[a,b]$, and $J_i(D)$, $i=1,2,3$ are local operators with constant coefficients. (Local operators with constant coefficients are differential operators of infinite order in the theory of hyperfunctions; see, e.g., [1], § 2. On the operation of $J_i(D)$, the measures μ_i are considered as hyperfunctions.)

We prepare two lemmas. Let $\mathscr{B}[K]$ denote the space of hyperfunctions with support in K. Let $H_K(\zeta)$ denote the supporting function $\sup_{x\in K}\operatorname{Re}\langle x, i\zeta\rangle$ of K ($i=\sqrt{-1}$).

Lemma 2. *The Fourier transform $\tilde{u}(\zeta)$ of $u\in\mathscr{B}[K]$ is an entire function which satisfies the following growth condition:*

$$|\tilde{u}(\zeta)|\leqslant C\exp(|\zeta|/\varphi(|\zeta|)+H_K(\zeta)),$$

where $\varphi(r)$ is a monotonely increasing function of $r\geqslant 0$ and satisfies $\varphi(0)=1$, $\varphi(r)\to\infty$ when $r\to\infty$.

Proof. The following estimate for $\tilde{u}(\zeta)$ is well known:

$$|\tilde{u}(\zeta)|\leqslant C_\varepsilon\exp(\varepsilon|\zeta|+H_K(\zeta))\qquad\text{for any } \varepsilon>0.$$

Put

$$\psi(r)=\sup_{|\zeta|=r}|\tilde{u}(\zeta)\exp(-H_K(\zeta))|\quad\text{and}\quad\psi_1(r)=r/\log(e+\psi(r)).$$

From the above estimates it is easily seen that $\psi_1(r)\to\infty$ when $r\to\infty$. Thus the function $\varphi(r)=\max\{\inf_{s\geqslant r}\psi_1(r),1\}$ serves our purpose. q.e.d.

Lemma 3. *Assume that the function $\varphi(r)$ has the properties mentioned in Lemma 2. Then for any prescribed constants A, C, c_1, c_2 there exists a local operator $J(D)$ whose Fourier transform $J(\zeta)$ satisfies the following estimate from below:*

$$|J(\zeta)|\geqslant C\exp(A|\zeta|/\varphi(|\zeta|))\qquad \textit{for } |\operatorname{Im}\zeta|\leqslant c_1+c_2|\operatorname{Re}\zeta|.$$

---

*) Partially supported by Fûjukai.

Such a function $J(\zeta)$ can be given by the infinite product

$$J(\zeta)=C'\prod_{m=1}^{\infty}\left(1+\frac{\zeta^2}{(m\varphi(m))^2}\right).$$

We omit the precise estimate (see [1], the proof of Lemma 6).

Remark. The $J(D)$ in this lemma is an elliptic local operator (see [2]).

Proof of Theorem 1. For given u, choose a function $\varphi(r)$ and a local operator $J(D)$ as in the lemmas above, and put $J_1(\zeta)=(\zeta+i)J(\zeta)$. Consider a Fourier hyperfunction U (see [2]) whose defining function is $\tilde{u}(\zeta)$ for $\operatorname{Im}\zeta>0$ and 0 for $\operatorname{Im}\zeta<0$. (Indeed U is the Fourier transform of the hyperfunction u considered as a Fourier hyperfunction.) U is divisible by $J_1(\zeta)$. In fact, by the estimate of $J_1(\zeta)$ from below, there exists a Fourier hyperfunction V whose defining function is $V_+(\zeta)=\tilde{u}(\zeta)/J_1(\zeta)$ for $\operatorname{Im}\zeta>0$ and $V_-(\zeta)=0$ for $\operatorname{Im}\zeta<0$. Thus $J_1(\zeta)V=U$. The inverse Fourier transform $\hat{V}$ of V is again a Fourier hyperfunction and satisfies (see the general theory of [2])

$$J_1(D)\hat{V}=\hat{U}=u.$$

On the other hand, the defining function $V_\pm$ of V has the estimate

$$|V_\pm(\zeta)|\leqslant C/(1+|\zeta|) \qquad \text{for } |\operatorname{Im}\zeta|\leqslant C.$$

Therefore the boundary values of those holomorphic functions define indeed an element of $L^2(R)$. The inverse Fourier transform $\hat{V}$ is also an element of $L^2(R)$.

Let μ_1 be the function in $L^2(R)$ which is equal to $\hat{V}$ on K and 0 otherwise. Then the hyperfunction $u-J_1(D)\mu_1$ has support only at the two points a, b. Thus by the structure theorem of hyperfunctions whose support is only one point, we have

$$u-J_1(D)\mu_1=J_2(D)\mu_2+J_3(D)\mu_3,$$

where μ_2 (respectively μ_3) is the Dirac measure at the point a (respectively at b). Thus $u=J_1(D)\mu_1+J_2(D)\mu_2+J_3(D)\mu_3$. q.e.d.

Remark. The same proof employing the theory of Fourier hyperfunctions shows that also in the case of several variables each hyperfunction is locally equal to an infinite derivative of an L^2 function.

Remark. In analogy with the case of distributions one may expect that $u\in\mathscr{B}[K]$ can be expressed by two terms: $u=J_1(D)\mu_1+J_2(D)\mu_2$. But this cannot be seen from the above method of proof. We can explain this difficulty in the following way: Since the differential operator of infinite order is not in general simultaneously elliptic and hyperbolic, the regularization usually causes the propagation of support to the whole space. It is conjectured by Sato, however, that we may always reduce the expression to two terms.

Theorem 4. *Let $\{u_k\}_{k=1}^{\infty}\subset\mathscr{B}[K]$ be a sequence which converges to $u\in\mathscr{B}[K]$ in the usual (Fréchet space) topology of $\mathscr{B}[K]$. Then, there is a local operator $J(D)$ with constant coefficients and a sequence of* *functions* $\{f_k\}_{k=1}^{\infty} \subset L^2(R)$ *which converges to* $f \in L^2(R)$, *such that* $J(D)f = u$, $J(D)f_k = u_k$.

Proof. It is easy to see that the estimates of the type in Lemma 2 hold for elements in a compact subset of $\mathscr{B}[K]$ with a common function $\varphi(r)$. Let $J_1(\zeta)$ be the entire infra-exponential function constructed in Lemma 3 with the φ corresponding to the compact subset $\{\tilde{u}, \tilde{u}_k, k=1, 2, \cdots\}$ of $\mathscr{B}[K]$. As in the proof of Theorem 1, the sequence $\{\tilde{u}_k(\zeta)/J_1(\zeta)(\zeta+i)\}_{k=1}^{\infty}$ is a bounded subset of $L^2(R)$. On the other hand, $\tilde{u}_k(\zeta) \exp(-\varepsilon|\zeta|)$ converges for any $\varepsilon > 0$ uniformly on the real line by the assumption. Thus $\{\tilde{u}_k(\xi)/J_1(\xi)(\xi+i)\}_{k=1}^{\infty}$ converges to $\tilde{u}(\xi)/J_1(\xi) \cdot (\xi+i)$ uniformly on every compact subset of R. Therefore $\tilde{u}(\xi)/J_1(\xi) \cdot (\xi+i)^2$ converges to $\tilde{u}(\xi)/J_1(\xi)(\xi+i)^2$ in $L^2(R)$. Applying the inverse Fourier transform, we obtain the desired f_k, f, and $J(D) = J_1(D)(D+i)^2$.

q.e.d.

For a compact set $K \subset \mathbf{R}^n$, $\mathscr{A}(K)$ denotes the space of real analytic functions on a neighborhood of K. This is a DFS (dual Fréchet Schwartz) space with its natural inductive limit topology.

Lemma 5. *The following are equivalent.*

1) *The weak topology* $(\sigma(A(K), B[K]))$ *of* $\mathscr{A}(K)$ *relative to the pairing with* $\mathscr{B}[K]$ *is weaker than the topology of* $\mathscr{A}(K)$ *defined by the seminorms* $\|u\|_J = \sup_{x \in K} |J(D)u(x)|$, *where* $J(D)$ *runs over the local operators with constant coefficients.* (*In the sequel the space* $\mathscr{A}(K)$ *endowed with the latter topology is denoted by* $\mathscr{A}_J(K)$.)

2) *Every hyperfunction* $u \in \mathscr{B}[K]$ *has an expression of the form* $u = J_1(D)\mu_1 + \cdots + J_N(D)\mu_N$, *where* μ_i *are measures with support in* K.

Proof. 1)$\Rightarrow$2). Take $u \in \mathscr{B}[K]$ arbitrarily. By the assumption we have the following inequality

$$|\langle f, u\rangle| \leqslant \sup_{x \in K} |J_1(D)f(x)| + \cdots + \sup_{x \in K} |J_N(D)f(x)|, \qquad f \in \mathscr{A}(K),$$

where $J_i(D)$, $i=1, \cdots, N$ are some local operators. Therefore u is a linear functional on the subspace

$$\left\{\begin{pmatrix} J_1(D)f \\ \vdots \\ J_N(D)f \end{pmatrix};\ f \in \mathscr{A}(K)\right\} \subset C(K)^N,$$

and u is continuous for the usual topology of $C(K)^N$ defined by the supremum norm on K. Hence by the Hahn-Banach theorem u can be continuously extended to the whole $C(K)^N$ and by the Riesz theorem it can be expressed as follows:

$$\begin{aligned} \langle f, u\rangle &= \int_K J_1(D)f\mu_1 + \cdots + \int_K J_N(D)f\mu_N \\ &= \langle f, J_1(-D)\mu_1 + \cdots + J_N(-D)\mu_N\rangle. \end{aligned}$$

Thus

$$u=J_1(-D)\mu_1+\cdots+J_N(-D)\mu_N,$$

where μ_i are measures with support in K. q.e.d.

2)⇒1) is easy.

From this lemma and Theorem 1 we have

Corollary 6. *Let* $K=[a, b]\subset \boldsymbol{R}^1$. *Then the assertion* 1) *of Lemma* 5 *holds.*

References

[1] Kaneko, A.: On continuation of regular solutions of partial differential equations to compact convex sets. J. Fac. Sci. Univ. Tokyo, Sec. IA, **17**, 567–580 (1970).

[2] Kawai, T.: On the theory of Fourier hyperfunctions and its application to partial differential equations with constant coefficients. Ibid., 467–517.

[3] Kaneko, A.: On the representation of hyperfunctions by measures. Proceedings of Symposium on Hyperfunctions and Partial Differential Equations at RIMS, March, 1971, Kyoto Univ., **145**, 92–108.