128. *On Some Separation Properties of a Function Algebra*

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§1. In this paper we shall consider the relation among some separation property of a function algebra on a compact Hausdorff space and a property of a representing space of a strongly regular function algebra. According to the first one, we can eliminate the assumption "in a weak sense" from "approximately normal in a weak sense" and "approximately regular in a weak sense" of the theorem in the previous paper [5]. Recently D. Wilken [10] has shown that there exists no strongly regular function algebra on the closed unit interval I except $\mathcal{C}$ [I], according to the second one, we shall obtain a sufficient condition for a representing space $\mathcal{X}$ to satisfy that any strongly regular function algebra on $\mathcal{X}$ is nothing but $\mathcal{C}(\mathcal{X})$ itself.

Throughout this paper let $\mathcal{M}(\mathcal{A})$ be the maximal ideal space of a function algebra $\mathcal{A}, \Gamma(\mathcal{A})$ the Silov boundary and Cho $(\mathcal{A})$ the Choquet boundary of $\mathcal{A}$, respectively. Further, let, for any subset S of $\mathcal{X}$ (or $\mathcal{M}(\mathcal{A})$), $f(S)$ be the set $\{f(x)\,;\, x \in S\}$, $\mathcal{A}|S$ the restriction of $\mathcal{A}$ to S and A_s the uniform closure of $\mathcal{A}|S$ in $\mathcal{C}(S)$.

§2. We owe the following definition to D. Wilken [9].

Definition. A function algebra $\mathcal{A}$ is said to be approximately regular on $\mathcal{X}$, iff, for each point p in X and each closed set K not containing p and for any positive number ε, there is a function f in $\mathcal{A}$ such that $f(p)=1$ and $|f(y)<\varepsilon$ for y in K. $\mathcal{A}$ is said to be approximately normal on $\mathcal{X}$ iff, for any two disjoint closed subsets K_1 and K_2 and for any $\varepsilon>0$, there is a function f in $\mathcal{A}$ such that $|f(x)-1|<\varepsilon$ on K_1 and $|f(y)|<\varepsilon$ on K_2.

Let us define a new separation property of a function algebra as follows.

Definition. A function algebra $\mathcal{A}$ satisfies the condition $(*)$ on a closed subset S of $\mathcal{M}(\mathcal{A})$ iff for any connected closed subset K in S, the Silov boundary of $\mathcal{A}_K$ is K.

It is evident that if $\mathcal{A}$ is approximately normal, then $\mathcal{A}$ is approximately regular and if $\mathcal{A}$ is approximately regular, then $\mathcal{A}$ satisfies the condition $(*)$. We know by the following example that, in general, $(*)$ is weaker than approximate regularity [5].

Example. Let $\mathcal{X}=\{z\,;\,|z|\leqq 1\}$, $T=\{z\,;\,|z|=1\}$, $\mathcal{A}=\{f \in \mathcal{C}(\mathcal{X})\,;$ for the restriction of f to T, there is a function $\tilde{f}$ which is analytic in $\mathcal{X}-T$ and $f(0)=\tilde{f}(0)\}$.

In spite of the weakness, as we shall show it, on the maximal ideal space these three concepts are equivalent.

Theorem. *Let $\mathcal{A}$ be a function algebra on $\mathcal{X}$. If $\mathcal{A}$ satisfies the condition $(*)$ on $\mathcal{M}(\mathcal{A})$, then $\mathcal{A}$ is approximately normal on $\mathcal{M}(\mathcal{A})$.*

To prove the theorem we shall use two lemmas without proof.

Lemma 1. *Let F be an intersection of peak sets of $\mathcal{A}$ in $\mathcal{M}(\mathcal{A})$. Then $\mathcal{A}|F$ is closed in $C(F)$ and $\mathcal{M}(\mathcal{A}|F)=F$ and $\Gamma(\mathcal{A}|F)\subset\Gamma(\mathcal{A})\cap F$* [8].

A subset K is called an *antisymmetric set* of $\mathcal{A}$ iff f in $\mathcal{A}$ takes always real values on K, then it takes only a single value on K. $\mathcal{A}$ is called antisymmetric algebra on $\mathcal{X}$ iff $\mathcal{X}$ itself is an antisymmetric set.

Let $\mathcal{P}$ be the collection of all maximal antisymmetric sets in $\mathcal{M}(\mathcal{A})$ which consist of a single point. E. Bishop [1] and I. Gricksberg [2] have obtained the decomposition of a function algebra into antisymmetric algebras and J. Tomiyama [8] called our attension to the fact that the decomposition of a representing space always deduced from that of $\mathcal{M}(\mathcal{A})$ and the set $\mathcal{P}$ defined above plays a special role.

Lemma 2. *Let $\mathcal{M}(\mathcal{A})=\mathcal{P}\cup\mathcal{K}_\alpha\cup\mathcal{K}_\beta\cup\cdots$ be the decomposition of $\mathcal{M}(\mathcal{A})$ into the maximal antisymmetric parts for $\mathcal{A}$. Then each part $\mathcal{K}_\alpha$ is connected and, for any representing space $\mathcal{X}$, $\mathcal{X}=\mathcal{P}\cup(\mathcal{K}_\alpha\cap\mathcal{X})\cup(\mathcal{K}_\beta\cap\mathcal{X})\cup\cdots$ is the decomposition of $\mathcal{X}$ into the maximal antisymmetric parts and further the set $\mathcal{P}$ is invariant, that is, the collection of all maximal antisymmetric sets in $\mathcal{X}$ consisting of a single point coincides with $\mathcal{P}$* [8].

Proof of the theorem. By using the Silov's theorem [7] we can easily prove that the following conditions are equivalent;

1. $\mathcal{A}$ is approximately normal on $\mathcal{M}(\mathcal{A})$.

2. For any closed subset F of $\mathcal{M}(\mathcal{A})$, the maximal ideal space $\mathcal{M}(\mathcal{A}_F)$ is F, i.e., $\mathcal{A}$ is a convex.

To prove our theorem we have only to prove 2. Suppose 2 were not true, that is, there were a closed subset F such that the set $\mathcal{M}(\mathcal{A}_F)$ contains F properly. Then there would be a point w in $\mathcal{M}(\mathcal{A}_F)-F$. Let the antisymmetric decomposition of $\mathcal{M}(\mathcal{A}_F)$ be $\mathcal{P}\cup\mathcal{K}_\alpha\cup\mathcal{K}_\beta\cup\cdots$. As F is a representing space of $\mathcal{A}_F$, F is decomposed into the following form; $F=\mathcal{P}\cup(\mathcal{K}_\alpha\cap F)\cup(\mathcal{K}_\beta\cap F)\cup\cdots$. Now there would be a maximal antisymmetric set $\mathcal{K}_0$ containing w, connected and not a single point. Let us notice that w is in $\mathcal{K}_0$ and not in $\mathcal{K}_0\cap F$. Since the set $\mathcal{K}_0$ is connected closed in $\mathcal{M}(\mathcal{A}_F)$ and $\mathcal{M}(\mathcal{A}_F)$ is closed in $\mathcal{M}(\mathcal{A})$, $\mathcal{K}_0$ is connected closed in $\mathcal{M}(\mathcal{A})$. As $\mathcal{A}$ satisfies $(*)$ on $\mathcal{M}(\mathcal{A})$, $\Gamma(\mathcal{A}_{\mathcal{K}_0})=\mathcal{K}_0$. A maximal antisymmetric set is a intersection of peak sets of $\mathcal{A}_F$.

Therefore we have the relation as follows;

$$\mathcal{K}_0=\Gamma(\mathcal{A}_{\mathcal{K}_0})=\Gamma(\mathcal{A}_F\,|\,\mathcal{K}_0)\subset\Gamma(\mathcal{A}_F)\cap\mathcal{K}_0\subset F\cap\mathcal{K}_0.$$

This contradicts the assumption $\mathcal{K}_0\cap F\subsetneqq\mathcal{K}_0$. Accordingly we know that $\mathcal{A}$ is approximately normal on $\mathcal{M}(\mathcal{A})$.

Corollary 1. *If* (1) Cho $(\mathcal{A})=\mathcal{M}(\mathcal{A})$ *or* (2) $\mathcal{A}$ *is maximal on* $\mathcal{M}(\mathcal{A})$, *then* $\mathcal{A}$ *is approximately normal* [9].

Proof. It is evident that if (1) holds then $(*)$ is satisfied on $\mathcal{M}(\mathcal{A})$. If (2) holds, then $\Gamma(\mathcal{A})=\mathcal{M}(\mathcal{A})$. Since, for any closed connected set F, $\mathcal{A}_F$ is also maximal on F or $C(F)$, $\Gamma(\mathcal{A}_F)=F$, that is, $(*)$ is satisfied on $\mathcal{M}(\mathcal{A})$. Therefore $\mathcal{A}$ is approximately normal on $\mathcal{M}(\mathcal{A})$.

Corollary 2. *Let* $\mathcal{A}$ *be a function algebra on* I *(or* T*) with* $\mathcal{M}(\mathcal{A})=I$ *(or* T*). Then* $\mathcal{A}$ *is approximately normal* [9].

Proof. We shall present the proof only for the interval; the proof on the circle is essentially the same.

Any closed connected subset F of I is again a closed interval. H. Rossi [6] has shown that if $\mathcal{M}(\mathcal{A})=I$, then $\Gamma(\mathcal{A})=I$. For any closed interval F, $\mathcal{A}_F$ is the function algebra on F with $\mathcal{M}(\mathcal{A}_F)=F$. Again applying the theorem of H. Rossi to $\mathcal{A}_F$, we obtain $\Gamma(\mathcal{A}_F)=F$. Therefore $\mathcal{A}$ satisfies the condition $(*)$ on $\mathcal{M}(\mathcal{A})$, whence $\mathcal{A}$ is approximately normal.

Let us notice that $\mathcal{A}$ satisfies $(*)$ on $\mathcal{M}(\mathcal{A})$, then $\Gamma(\mathcal{A})=\mathcal{M}(\mathcal{A})$, but when $\mathcal{X}$ is not connected, even if A satisfies $(*)$ on X, $\Gamma(A)$ is not in general $\mathcal{X}$ and conversely, $\Gamma(\mathcal{A})=\mathcal{M}(\mathcal{A})$ does not imply $(*)$ on $\mathcal{M}(\mathcal{A})$ [4], [9].

§3. Definition. A function algebra $\mathcal{A}$ on $\mathcal{X}$ is said to be "strongly regular on $\mathcal{X}$" iff it has the following property; for each f in A, each point x in X and every $\varepsilon>0$, there is a neighbourhood U of x and a function g in $\mathcal{A}$ with $g(y)=f(x)$ for all y in U and $\|f-g\|<\varepsilon$.

D. Wilken pointed out that a strongly regular function algebra was normal. Therefore if $\mathcal{A}$ is strongly regular on $\mathcal{X}$, then $\mathcal{X}$ is $\mathcal{M}(\mathcal{A})$ and a function in $C(\mathcal{X})$ locally approximable by $\mathcal{A}$ belongs to $\mathcal{A}$.

Theorem. *If* $\mathcal{A}$ *is a strongly regular function algebra on* I, *then* $\mathcal{A}=C(I)$.

This theorem has been proved by D. Wilken [10]. He said in the paper that it has been conjectured that any strongly regular function algebra must coincide with $C(\mathcal{X})$, but it would be interesting to construct a counter example. Likewise it would be interesting to characterize the space $\mathcal{X}$ on which any strongly regular function algebra coincides with $C(\mathcal{X})$.

In this paper we shall show that any strongly regular function algebra is $C(\mathcal{X})$ on the space $\mathcal{X}$ with the following property.

Property. The space $\mathcal{X}$ is a C_δ-space* and for any disjoint two points x, y, there are two closed subsets F_1, F_2 such that $x \in F_1$, $y \in F_2$, $F_1 \cup F_2 = \mathcal{X}$ and $F_1 \cap F_2 = \phi$ or $\{p\}$. (a point p is Cho(A) and is neither x nor y.)

So far as considering the strong regularity of $\mathcal{A}$, unit interval and a compact totally ordered G_δ-space with interval topology are the special cases of the space with the above property.

Let us notice that the property does not necessarily mean the ordered relation in $\mathcal{X}$. For example, if $\mathcal{X}$ is a totally diaconnected compact G_δ-space, then $\mathcal{X}$ has the above property.

Theorem. *Let $\mathcal{X}$ be a space with the above property and $\mathcal{A}$ strongly regular on $\mathcal{X}$. Then $\mathcal{A} = C(\mathcal{X})$.*

Proof. To prove $\mathcal{A} = C(\mathcal{X})$ it is enough to show that for any closed subset F in $\mathcal{X}$, $\mathcal{A}|F$ is closed in $C(F)$ [3].

Let x, y be disjoint two points. Then there are two closed subset F_1, F_2 with the above property. We shall distinguish here two cases, i.e., $F_1 \cap F_2 = \phi$ and $= \{p\}$. For the first, as $\mathcal{X} = \mathcal{M}(\mathcal{A})$, the characteristic function χ_{F_1}, χ_{F_2} are in $\mathcal{A}$ by the Sliov's theorem [7]. Thus F_1 and F_2 are peak sets which contain x and y respectively as interior point. For the second, we have only to consider the case where p is not an isolated point, since if p is an isolated point, then the case is reduced to the first one. As p is in Cho$(\mathcal{A})$, for any neighbourhood U of p and positive number ε, there is a function f in $\mathcal{A}$ such that $f(p) = 1$ $(= \|f\|)$ and $|f(z)| < \varepsilon$ for all z in $X - U$. Let g be the function satisfying that $g(z) = 1$ on F_1 and $g(z) = f(z)$ on F_2. By the same methods of D. Wilken [10], we know that g is in A. Therefore, as the first case, we have shown that there is a peak set which contains x (or y) as an interior point and does not contain y (or x). Hence we can easily show [3] that for any closed set F in X, $\mathcal{A}|F$ is closed in $C(F)$. Accordingly $\mathcal{A} = C(\mathcal{X})$.

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*) A G_δ-space is a space that each point is a G_δ-set.

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