4. On the Integration of the Temporally Inhomogeneous Diffusion Equation in a Riemannian Space

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1. Introduction. Let R be a connected domain of an m-dimensional, orientable C^∞ Riemann space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$. We consider the forward diffusion equation in R

$$(1.1)\qquad E_{tx}f = \frac{\partial f(t, x)}{\partial t} - A_{tx}f(t, x) = 0, \ t>s,$$

where

$$(1.2)\qquad A_{tx}f(t, x) = g(x)^{-1/2}\frac{\partial^2}{\partial x^i \partial x^j}(g(x)^{1/2}a^{ij}(t, x)f(t, x))$$
$$-g(x)^{-1/2}\frac{\partial}{\partial x^i}(g(x)^{1/2}b^i(t, x)f(t, x)) + c(t, x)f(t, x),$$
$$g(x) = \det(g_{ij}(x)).$$

The associated backward diffusion equation is defined by

$$(1.3)\qquad E^*_{sy}h = -\frac{\partial h(s, y)}{\partial s} - A^*_{sy}h(s, y) = 0, \ s<t,$$

where A^*_{sy} is the formal adjoint of A_{ty}:

$$(1.4)\qquad A^*_{sy}h(s,y) = a^{ij}(s,y)\frac{\partial^2 h(s,y)}{\partial y^i \partial y^j} + b^i(s,y)\frac{\partial h(s,y)}{\partial y^i} + c(s,y)h(s,y).$$

The operator $A_t = A_{tx}$ is assumed to be elliptic in x in the sense that

$$(1.5)\qquad a^{ij}(t,x)\xi_i\xi_j > 0 \ \text{for} \ \sum_i (\xi_i)^2 > 0.$$

Since the value of $A_{tx}f(t,x)$ should be independent of the local coordinates $(x^1, \ldots, x^m)$, we must have, by the coordinates change $x \to \bar{x}$, the transformation rule

$$(1.6)\qquad \bar{a}^{ij}(t,\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^n}a^{kn}(t,x),$$
$$\bar{b}^i(t, \bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k}b^k(t, x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^n}a^{kn}(t, x).$$

For the sake of simplicity, we assume that the coefficients $a^{ij}(t, x)$, $b^i(t, x)$, $c(t, x)$ and $g_{ij}(x)$ are C^∞ functions of (t, x).

The purpose of the present note is to give a sketch of a method[1)]

1) Another method was proposed by Tosio Kato: (Integration of the equation of evolution in a Banach space, J. Math. Soc. Japan, **5**, 208–234 (1953)). His method is much general and elegant. However, it may not be easy to apply his method to the concrete equation such as (1.1), since he assumes that the domain $D(\bar{A}_t)$ of the closed extention $\bar{A}_t$ of A_t is independent of t.

of the integration in the function space $L=L_1(R)$ [2] of (1. 1) with the initial condition

(1. 7) $$f(s, x)=f(x)\,.$$

Thus we firstly consider an approximate equation

(1. 1)′ $$E_{tx}^{(n)}f^{(n)}=\frac{\partial f^{(n)}(t, x)}{\partial t}-A_{tx}^{(n)}f^{(n)}(t, x)=0,\ t\geqq s\ \ (n=1, 2, \ldots),$$
$$f^{(n)}(s, x)=f(x)\,,$$

where $A_t^{(n)}=A_{tx}^{(n)}$ is a bounded operator in L which converges, in the sense to be explained below, to $A_t=A_{tx}$ as $n\to\infty$. Secondly it will be shown that there exists a subsequence $\{f^{(n')}(t, x)\}$ of the solutions $\{f^{(n)}(t, x)\}$ of (1. 1)′ which converges, in the sense of the "distribution",[3] to a solution T_t of

(1. 1)″ $$\frac{\partial(T_t, \varphi)}{\partial t}-(T_t, A_t^*\varphi)=0,\ t>s\,,$$
$$(T_s, \varphi)=(f, \varphi)=\int_R f(x)\varphi(x)dx\,.$$

Here (T_t, φ) is the value of the distribution T_t at φ, $\varphi(x)$ denoting a C^∞ function whose carrier is compact and is contained in an open domain of R. The totality of such functions $\varphi(x)$ will be denoted by $D(R)$. Finally we will show, by a parametrix consideration, that this T_t is defined by a "genuine" solution of (1.1). (See the Theorem below in 3.)

2. The Construction of the Distribution T_t. Let D be a set of C^∞ functions $f(x)$ with compact carriers such that D is L-dense in L. We regard $A_t=A_{tx}$ as an additive operator defined on $D\subseteqq L$ to L. Let $\bar{A}_t$ be the smallest closed extension of the operator A_t, and we will make the following:

Hypothesis.[4] Let, for all sufficiently large integer n (independently of t), the resolvents

(2. 1) $$I_t^{(n)}=(I-n^{-1}\bar{A}_t)^{-1}$$

exist as bounded operators on L to L such that

(2. 2) $I_t^{(n)}f(x)$ is non-negative and $\int_R I_t^{(n)}f(x)dx=\int_R f(x)dx$

if $f(x)\in L$ is non-negative,

(2. 3) $I_t^{(n)}f$ is strongly continuous in t.

The Hypothesis implies that

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2) The function space of the Borel measurable functions $f(x)$ which are integrable with respect to the measure $dx=g(x)^{1/2}\,dx^1\ldots dx^m$. The norm of f is hence given by $\|f\|=\int_R|f(x)|\,dx$. It is to be noted that our method of integration of (1.1) may, with slight modifications, be extended to the case of the function space $L_p(R)$, $1\leqq p\leqq\infty$.

3) L. Schwartz: Théorie des distributions, I et II, Paris (1950 et 1951).

4) Cf. K. Yosida: On the integration of diffusion equations in Riemannian spaces, Proc. Amer. Math. Soc., **3**, 864–873 (1952).

(2. 4) $$\|I_t^{(n)}\| \leqq 1$$
and
(2. 5) $$A_t^{(n)} = \overline{A}_t I_t^{(n)} = n(I_t^{(n)} - I)$$
satisfies
(2. 6) strong $\lim_{n\to\infty} A_t^{(n)} f = \overline{A}_t f$ for f in the domain $D(\overline{A}_t)$ of $\overline{A}_t$.

We first prove the

Lemma 1. For any $f \in L$, there exists a solution $f_{ts}^{(n)} = f^{(n)}(t, s, x) \in L$ of
(2. 7) $$D_t f_{ts}^{(n)} = \text{strong}\lim_{\delta\to 0} \delta^{-1}(f_{t+\delta, s}^{(n)} - f_{ts}^{(n)}) = A_t^{(n)} f_{ts}^{(n)},\ t \geqq s,$$
$$\text{strong}\lim_{t\to s} f_{ts}^{(n)} = f,$$
satisfying
(2. 8) $f^{(n)}(t, s, x)$ is non-negative and $\int_R f^{(n)}(t, s, x)dx = \int_R f(x)dx$ if $f(x)$ is non-negative.

Proof. Putting
$$P^{(n)}(t, s) = \exp((t-s)A_s^{(n)}) = \sum_{k=0}^{\infty} (k!)^{-1}(n(t-s)(I_s^{(n)} - I))^k,$$
we have the bounded operator
$$W^{(n)}(t, s) = E_t^{(n)} P^{(n)}(t, s) = (A_t^{(n)} - A_s^{(n)})P^{(n)}(t, s).$$
Then the solution $f_{ts}^{(n)}$ of (2. 7) may be defined by
(2. 9) $$f_{ts}^{(n)} = P^{(n)}(t, s)f - \int_s^t P^{(n)}(t, \tau)Q^{(n)}(\tau, s)f\, d\tau,$$
where
(2. 10) $$Q^{(n)}(t, s) = \sum_{m=1}^{\infty} (-1)^{m+1} W_m^{(n)}(t, s),$$
$$W_m^{(n)}(t, s) = \int_s^t W^{(n)}(t, \tau) W_{m-1}^{(n)}(\tau, s)d\tau,\ W_1^{(n)} = W^{(n)}.$$
We next prove
(2. 11) $$\|f_{ts}^{(n)}\| \leqq \|f\|.$$
For this purpose, we start, by (2. 5) and (2.7),
(2. 12) $$f_{t+\delta,s}^{(n)} = f_{ts}^{(n)} + \delta(n(I_t^{(n)} - I)f_{ts}^{(n)}) + o(\delta),\ \delta > 0.$$
Then, by (2. 4),
$$\|f_{t+\delta,s}^{(n)}\| \leqq (1-\delta n)\|f_{ts}^{(n)}\| + \delta n\|f_{ts}^{(n)}\| + o(\delta) \leqq \|f_{ts}^{(n)}\| + o(\delta),$$
and hence
$$\frac{d^+\|f_{ts}^{(n)}\|}{dt} \leqq 0$$
which proves (2. 11). We next assume that $f^{(n)}(t, s, x)$ to be non-negative. Then, by (2. 2) and (2. 12),
$$\int_R f^{(n)}(t+\delta, s, x)dx = \int_R f^{(n)}(t, s, x)dx + \int_R \delta n(I_t^{(n)} - I)f^{(n)}(t, s, x)dx$$
$$+ \int_R o(\delta)dx \geqq \int_R f^{(n)}(t, s, x)dx + o(\delta),$$
which implies

(2.13) $$\frac{d^+}{dt}\int_R f^{(n)}(t, s, x)dx \geqq 0\,.$$

Hence, if $f(x)$ is non-negative, we have, by (2.11) and (2.13),

$$\|f_{ts}^{(n)}\| = \int_R |f^{(n)}(t, s, x)|\,dx \leqq \|f\| = \int_R |f(x)|\,dx$$
$$= \int_R f(x)dx \leqq \int_R f^{(n)}(t, s, x)dx\,.$$

Therefore $f^{(n)}(t, s, x)$ must be non-negative (almost everywhere) with $f(x)$.

Lemma 2. There exists a subsequence $\{f_{ts}^{(n')}\}$ of $\{f_{ts}^{(n)}\}$ such that

(2.14) $$\lim_{n'\to\infty}(f_{ts}^{(n')}, \varphi) = (T_t, \varphi),\ \ \varphi \in D(R)\,,$$

where (T_t, φ) satisfies (1.1)$''$. Actually, T_t is a distribution defined by a measure ρ_{ts}:

(2.15) $$(T_t, \varphi) = \int_R \varphi(x)d\rho_{ts}(x)\,.$$

Proof. Integrating (2.7), we have

$$(f_{ts}^{(n)}, \varphi)-(f, \varphi) = \int_s^t (A_\tau^{(n)}f_{\tau s}^{(n)}, \varphi)d\tau = \int_s^t (I_\tau^{(n)}f_{\tau s}^{(n)}, A_\tau^*\varphi)d\tau\,,$$

and hence, by (2.4) and (2.11),

(2.16) $$|(f_{t_1,s}^{(n)}-f_{ts}^{(n)}, \varphi)| \leqq \|f\|\cdot\left|\int_t^{t_1}\max_x |A_{\tau x}^*\varphi(x)|\,d\tau\right|.$$

By virtue of (2.11) and (2.16), we may choose a subsequence $\{n'\}$ of $\{n\}$ such that (2.14) holds for a distribution T_t which satisfies

(2.17) (T_t, φ) is continuous in t.

We see that (2.15) also holds good by (2.8). We have also

(2.18) $$\lim_{n'\to\infty}(I_\tau^{(n')}f_{\tau s}^{(n')}, A_\tau^*\varphi) = (T_\tau, A_\tau^*\varphi) \text{ boundedly in } \tau,\ s \leqq \tau \leqq t\,,$$

since, by (2.11),

$$(I_\tau^{(n)}f_{\tau s}^{(n)}, \varphi)-(I_\tau^{(n)}f_{\tau s}^{(n)}, (I-n^{-1}A_\tau^*)\varphi) = n^{-1}(I_\tau^{(n)}f_{\tau s}^{(n)}, A_\tau^*\varphi)\,,$$
$$(I_\tau^{(n)}f_{\tau s}^{(n)}, (I-n^{-1}A_\tau^*)\varphi) = ((I-n^{-1}\bar{A}_\tau)(I-n^{-1}\bar{A}_\tau)^{-1}f_{\tau s}^{(n)}, \varphi) = (f_{\tau s}^{(n)}, \varphi),$$
$$n^{-1}|(I_\tau^{(n)}f_{\tau s}^{(n)}, A_\tau^*\varphi)| \leqq n^{-1}\|f\|\cdot\max_x|A_{\tau x}^*\varphi(x)|\,.$$

Thus T_t satisfies (1.1)$''$.

3. The Theorem. Let x_0 be any point of R and let $U(x_0)$ be a sufficiently small neighbourhood of x_0. Let $V(x_0)$ be any neighbourhood of x_0 such that its closure is contained in $U(x_0)$. We may construct[5] a parametrix $H(x, y, t, s)$ for the equation (1.3) such that

(3.1) $H(x, y, t, s)$ is, for $t>s$, C^∞ in (x, y, t, s),

(3.2) $E_{sx}^*H(x, y, t, s) = K(x, y, t, s)$ is C^∞ in (x, y, t, s) even when $t = s$,

(3.3) $H(x, y, t, s) \equiv 0$ if x or y is outside of $U(x)$,

(3.4) $$f(x) = \lim_{t\downarrow t_0, s\uparrow t_0}\int_R f(y)H(x, y, t, s)dy = \lim_{t\downarrow t_0, s\uparrow t_0}\int_R f(y)\,H(y, x, t, s)dy$$ for any $x \in V(x_0)$ and for any continuous function $f(y)$.

5) Cf. K. Yosida: On the fundamental solution of the parabolic equation in a Riemannian space, Osaka Math. J., **5**, 65–74 (1953).

We have thus, for any $\varepsilon>0$,

$$\begin{aligned}
&(f^{(n)}(t, s, x), H(x, y, t+\varepsilon, t))-(f^{(n)}(s, s, x), H(x, y, t+\varepsilon, s)) \\
&= \int_s^t \frac{d}{d\tau}(f^{(n)}(\tau, s, x), H(x, y, t+\varepsilon, \tau))d\tau \\
&= \int_s^t (E_{\tau x}^{(n)} f^{(n)}(\tau, s, x), H(x, y, t+\varepsilon, \tau))d\tau \\
&+ \int_s^t (f^{(n)}(\tau, s, x), -E_{\tau x}^* H(x, y, t+\varepsilon, \tau))d\tau \\
&+ \int_s^t \{(\overline{A}_{\tau x} I_\tau^{(n)} f^{(n)}(\tau, s, x), H(x, y, t+\varepsilon, \tau)) \\
&- (I_\tau^{(n)} f^{(n)}(\tau, s, x), A_{\tau x}^* H(x, y, t+\varepsilon, \tau))\}d\tau \\
&+ \int_s^t (I_\tau^{(n)} f^{(n)}(\tau, s, x)-f^{(n)}(\tau, s, x), A_{\tau x}^* H(x, y, t+\varepsilon, \tau))d\tau .
\end{aligned}$$

On the right hand side, the second term vanishes[6] in virtue of the Green's integral theorem and (3. 3). And the third term tends, as $n=n'\to\infty$, to zero. This we see by (2. 14) and (2. 18).

Therefore we have, for any $\varphi(y)\in D(R)$,

$$\begin{aligned}
\int_R d\rho_{ts}(x)(H(x, y, t+\varepsilon, t), \varphi(y)) &= \int_R f(x)(H(x, y, t+\varepsilon, s), \varphi(y))dx \\
&+ \int_s^t \{\int_R d\rho_{\tau s}(x)(K(x, y, t+\varepsilon, \tau), \varphi(y))\}d\tau .
\end{aligned}$$

By letting $\varepsilon\downarrow 0$ and remembering (3. 4), we obtain

$$\begin{aligned}
(3.\,5)\qquad \int_R d\rho_{ts}(x)\varphi(x) &= \int_R f(x)(H(x, y, t, s), \varphi(y))dx \\
&+ \int_s^t \{\int_R d\rho_{\tau s}(x)[\int_R K(x, y, t, \tau)\varphi(y)dy]\}d\tau .
\end{aligned}$$

The measure $\int_A d\rho_{ts}(x)$ is thus absolutely continuous with respect to the measure $\int_A dx$, and the density $f(t, s, x)$ satisfies, by (3. 5),

$$\begin{aligned}
(3.\,6)\qquad \int_R f(t, s, x)\varphi(x)dx &= \int_R f(x)(H(x, y, t, s), \varphi(y))dx \\
&+ \int_s^t \{\int_R d\rho_{\tau s}(x)[\int_R K(x, y, t, \tau), \varphi(y)]\}d\tau .
\end{aligned}$$

Hence $f(t, s, x)$, which satisfies

$$(3.\,7)\qquad (T_t, \varphi) = \int_R f(t, s, x)\varphi(x)dx ,$$

is equivalent to

$$(3.\,8)\qquad \int_R f(y)H(y, x, t, s)dy+\int_s^t \{\int_R d\rho_{\tau s}(y)K(y, x, t, \tau)\}d\tau .$$

This is surely continuously differentiable once in t and twice in x for $t>s$ and for $x\in V(x_0)$.

We have thus proved the following:

Theorem. Let the Hypothesis be satisfied. Then for any $f\in L$, there exists a solution $f_{ts}=f(t, s, x)\in L$ of (1. 1) with the initial condition

$$(3.\,9)\qquad \lim_{t\downarrow s} f(t, s, x)=f(x) \text{ almost everywhere.}$$

The uniqueness of this solution may be proved by the known argument. Moreover, $f(t, s, x)$ is non-negative with $f(x)$.

6) By a similar argument as in the paper by K. Yosida. Cf. 4), p. 870.