59. A Generalization of Ascoli's Theorem

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Let R be an abstract space. For a double system of mappings $\mathfrak{a}_{\gamma,\lambda}$ of R into uniform spaces S_λ $(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, there exists the weakest uniformity $\mathfrak{U}$ on R for which $\mathfrak{a}_{\gamma,\lambda}(\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$. In an earlier paper[1] we have obtained a condition for which R is complete by $\mathfrak{U}$. In this paper we shall consider conditions for which R is totally bounded by $\mathfrak{U}$ and as a generalization of Ascoli's theorem, we shall prove Theorem II which is essentially more general than that obtained by N. Bourbaki.[2]

Lemma 1. *Let* $\mathfrak{a}_\nu$ $(\nu=1, 2,\ldots, n)$ *be a finite number of mappings of* R *into uniform spaces* S_ν *with uniformities* $\mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ *respectively. If the image* $\mathfrak{a}_\nu(R)$ *is totally bounded in* S_ν *for every* $\nu=1, 2,\ldots, n$, *then for any* $U_\nu \in \mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ *we can find a finite number of points* $a_\mu \in R$ $(\mu=1, 2,\ldots, m)$ *such that*

$$R = \sum_{\mu=1}^{m} \prod_{\nu=1}^{n} \mathfrak{a}_\nu^{-1} U_\nu(a_\mu),$$

that is, for any $x \in R$ *we can find* μ *for which*

$$\mathfrak{a}_\nu(x) \in U_\nu(\mathfrak{a}_\nu(a_\mu)) \qquad \textit{for every } \nu=1, 2,\ldots, n.$$

Proof. For any $U_\nu \in \mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ we can find by definition $V_\nu \in \mathfrak{V}_\nu$ such that

$$V_\nu^{-1} \times V_\nu \leqq U_\nu \qquad (\nu=1, 2,\ldots, n).$$

Since the image $\mathfrak{a}_\nu(R)$ is totally bounded by assumption, we can find a finite number of points $y_{\nu,\mu} \in S_\nu$ $(\mu=1, 2,\ldots, m)$ such that

$$\mathfrak{a}_\nu(R) \subset \sum_{\mu=1}^{m_\nu} V_\nu(y_{\nu,\mu}) \qquad (\nu=1, 2,\ldots, n).$$

Corresponding to every system $\mu_\nu=1, 2,\ldots, m_\nu$ $(\nu=1, 2,\ldots, n)$ we select a point $a_{\mu_1\mu_2\ldots\mu_n} \in R$ such that

$$\mathfrak{a}_\nu(a_{\mu_1\mu_2\ldots\mu_n}) \in V_\nu(y_{\nu,\mu_\nu}) \qquad \text{for every } \nu=1, 2,\ldots, n,$$

if exists. Then for any $x \in R$ we can find μ_ν $(\nu=1, 2,\ldots, n)$ such that

$$\mathfrak{a}_\nu(x) \in V_\nu(y_{\nu,\mu_\nu}) \qquad \text{for every } \nu=1, 2,\ldots, n,$$

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1) H. Nakano: On completeness of uniform spaces, Proc. Japan Acad., **29**, 490–494 (1953).

2) N. Bourbaki: Topologie générale, **3**, Chap. 10, espaces fonctionnels. Paris (1949).

and we have obviously for every $\nu=1, 2, \dots, n$

$$V_\nu(y_{\nu,\mu_\nu}) \subset V_\nu^{-1} \times V_\nu(\mathfrak{a}_\nu(a_{\mu_1\mu_2\dots\mu_n})) \subset U_\nu(\mathfrak{a}_\nu(a_{\mu_1\mu_2\dots\mu_n})).$$

For a uniformly continuous mapping $\mathfrak{a}$ of a uniform space R into a uniform space S, we see easily by definition that if R is totally bounded, then the image $\mathfrak{a}(R)$ also is totally bounded in S. Thus, recalling the definition of weak uniformity, we obtain immediately by Lemma 1

Theorem I. *For a system of mappings $\mathfrak{a}_\lambda (\lambda \in \Lambda)$ of an abstract space R into uniform spaces $S_\lambda (\lambda \in \Lambda)$, the weak uniformity of R by $\mathfrak{a}_\lambda (\lambda \in \Lambda)$ is totally bounded if and only if the image $\mathfrak{a}_\lambda(R)$ is totally bounded in S_λ for every $\lambda \in \Lambda$.*

Lemma 2. *For an equi-continuous system of mappings $\mathfrak{a}_\lambda (\lambda \in \Lambda)$ of a uniform space R with uniformity $\mathfrak{U}$ into a uniform space S with uniformity $\mathfrak{V}$, if R is totally bounded by $\mathfrak{U}$ and the point set*

$$\{\mathfrak{a}_\lambda(x) \colon \lambda \in \Lambda\}$$

is totally bounded in S for every $x \in R$, then for any $U \in \mathfrak{V}$ we can find a finite number of elements $\lambda_\nu \in \Lambda (\nu=1, 2, \dots, n)$ such that for any $\lambda \in \Lambda$ we can find ν for which we have

$$\mathfrak{a}_\lambda(x) \in U(\mathfrak{a}_{\lambda_\nu}(x)) \qquad \text{for every } x \in R.$$

Proof. For any $U_0 \in \mathfrak{V}$ we can find by definition $V \in \mathfrak{V}$ such that

$$V \times V \times V \leqq U_0.$$

Since the system $\mathfrak{a}_\lambda (\lambda \in \Lambda)$ is equi-continuous by assumption, for such $V \in \mathfrak{V}$ we can find by definition a symmetric connector $U \in \mathfrak{U}$ for which $y \in U(x)$ implies $\mathfrak{a}_\lambda(y) \in V(\mathfrak{a}_\lambda(x))$ for every $\lambda \in \Lambda$. Since R is totally bounded by assumption, we can find by definition a finite number of points $x_\nu \in R$ $(\nu=1, 2, \dots, n)$ such that

$$R = \sum_{\nu=1}^{n} U(x_\nu).$$

Since the point set $\{\mathfrak{a}_\nu(x_\nu) \colon \lambda \in \Lambda\}$ is by assumption totally bounded for every $\nu=1, 2, \dots, n$, we can find by Lemma 1 a finite number of elements $\lambda_\mu \in \Lambda$ $(\mu=1, 2, \dots, m)$ such that for any $\lambda \in \Lambda$ we can find μ for which

$$\mathfrak{a}_\lambda(x_\nu) \in V(\mathfrak{a}_{\lambda_\mu}(x_\nu)) \qquad \text{for every } \nu=1, 2, \dots, n.$$

Then for any $x \in R$ we can find ν such that $x \in U(x_\nu)$ and we have

$$\begin{aligned} \mathfrak{a}_\lambda(x) &\in V(\mathfrak{a}_\lambda(x_\nu)) \subset V \times V(\mathfrak{a}_{\lambda_\mu}(x_\nu)) \\ &\subset V \times V \times V(\mathfrak{a}_{\lambda_\mu}(x)) \subset U_0(\mathfrak{a}_{\lambda_\mu}(x)), \end{aligned}$$

because $x \in U(x_\nu)$ implies $x_\nu \in U(x)$ and hence $\mathfrak{a}_{\lambda_\mu}(x_\nu) \in V(\mathfrak{a}_{\nu_\mu}(x))$.

Theorem II. *For a double system of mappings $\mathfrak{a}_{\gamma,\lambda}$ of an abstract space R into uniform spaces S_λ with uniformities $\mathfrak{V}_\lambda$ $(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, if the image $\mathfrak{a}_{\gamma,\lambda}(R)$ is totally bounded in S_λ for every $\gamma \in \Gamma_\lambda$ and*

$\lambda \in \Lambda$ and if for each $\lambda \in \Lambda$ we can find a totally bounded uniformity on the space Γ_λ for which the system of mappings $\mathfrak{a}_{\gamma,\lambda}(x) \in S_\lambda$ $(x \in R)$ of Γ_λ into S_λ is equi-continuous, then R is totally bounded by the weakest uniformity for which the system $\mathfrak{a}_{\gamma,\lambda}(\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$.

Proof. For each $\lambda \in \Lambda$ we denote by $\mathfrak{b}_\lambda$ the mapping of R into the power space $S_\lambda^{\Gamma_\lambda}$ with the power uniformity $\mathfrak{V}_\lambda^{\Gamma_\lambda}$ such that

$$\mathfrak{b}_\lambda(x) = (\mathfrak{a}_{\gamma,\lambda}(x))_{\gamma \in \Gamma_\lambda} \qquad \text{for every } x \in R.$$

Recalling Lemma 2, we obtain by assumption that for any $U_\lambda \in \mathfrak{V}_\lambda$ we can find a finite number of points $x_\nu \in R$ $(\nu = 1, 2, \ldots, n)$ such that for any $x \in R$ we can find ν for which we have

$$\mathfrak{a}_{\gamma,\lambda}(x) \in U(\mathfrak{a}_{\gamma,\lambda}(x_\nu)) \qquad \text{for every } \gamma \in \Gamma_\lambda,$$

that is, $\mathfrak{b}_\lambda(x) \in U^{\Gamma_\lambda}(\mathfrak{b}_\lambda(x_\nu))$. Thus we see that the image $\mathfrak{b}_\lambda(R)$ is totally bounded in S^{Γ_λ} by $\mathfrak{V}^{\Gamma_\lambda}$ for every $\lambda \in \Lambda$. Since the weak uniformity of R by $\mathfrak{b}_\lambda$ $(\lambda \in \Lambda)$ coincides with the weakest uniformity for which the system $\mathfrak{a}_{\gamma,\lambda}$ $(\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$, we conclude therefore Theorem II by Theorem I.