151. On Spaces Having the Weak Topology with Respect to Closed Coverings. II

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In the first paper under this title [4] we have introduced the following notion. Let X be a topological space and $\{A_\alpha\}$ a closed covering of X. Then X is said to *have the weak topology with respect to* $\{A_\alpha\}$, if the union of any subcollection $\{A_\beta\}$ of $\{A_\alpha\}$ is closed in X and any subset of $\bigcup_\beta A_\beta$ whose intersection with each A_β is open relative to the subspace topology of A_β is necessarily open in the subspace $\bigcup_\beta A_\beta$.

Any CW-complex (cf. [5]) has the weak topology with respect to the closed covering which consists of the closures[1] of all the cells. As another example we remark that a topological space has always the weak topology with respect to any locally finite closed covering.[2]

The purpose of this paper is to establish the following theorem.

Theorem 1. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. Then X is paracompact and normal if and only if each subspace A_α is paracompact and normal.*

Thus if X has the weak topology with respect to a closed covering $\{A_\alpha\}$, each of the following properties for all subspaces A_α implies the same property for X: (1) normality, (2) complete normality, (3) perfect normality, (4) collectionwise normality, (5) paracompactness and normality, (6) countable paracompactness and normality. On the other hand, local compactness or metrizability[3] for all A_α does not necessarily imply the same property for X.

§1. Lemmas

Lemma 1. *Let A be a closed subset of a paracompact and normal space X. If $\{G_\alpha\}$ is a locally finite system in A which consists of open F_σ-sets G_α of A, then there exists a locally finite system $\{H_\alpha\}$ of open F_σ-sets of X with the following properties:*

1) The closure of a cell e should be understood here as that in the complex, that is, as the intersection of all subcomplexes containing e.

2) From Theorem 1 below it follows immediately that a topological space which is the union of a locally finite collection of closed, paracompact, normal subspaces is paracompact and normal; this proposition is remarked also by E. Michael [2].

3) We have learned that the latter proposition given in the remark at the end of [4] was already proved by J. Nagata in his paper: On a necessary and sufficient condition of metrizability, Jour. Inst. Polytech. Osaka City Univ., Ser. A, **1**, 93–100 (1950).

(a) $\qquad G_\alpha = H_\alpha \frown A$ *for each* α,

(b) $\{H_\alpha\}$ *is similar to* $\{G_\alpha\}$; *i.e.* $\overset{r}{\underset{i=1}{\frown}} G_{\alpha_i}=0$ *implies* $\overset{r}{\underset{i=1}{\frown}} H_{\alpha_i}=0$.

Proof. By assumption for each α there exists a countable collection of closed sets $F_{\alpha n}$ of A (and hence of X) such that $G_\alpha = \overset{\infty}{\underset{n=1}{\smile}} F_{\alpha n}$. Since X is paracompact and $\{F_{\alpha 1}\}$ is locally finite in X, by [3, Theorem 1.3] there exists a system $\{H_{\alpha 1}\}$ of open F_σ-sets of X such that

$$(a_1) \qquad F_{\beta 1} \subset H_{\alpha 1},\ \overline{H}_{\alpha 1} \subset (G_\alpha \smile (X-A)) \frown V_\alpha,$$

$$(b_1) \qquad \{\overline{H}_{\alpha 1}\} \text{ is similar to } \{F_{\alpha 1}\}.$$

Here $\{V_\alpha\}$ is a locally finite system of open sets of X such that $\overline{G}_\alpha \subset V_\alpha$ for each α; the existence of such a system $\{V_\alpha\}$ is assured by [3, Lemma in §3] since $\{\overline{G}_\alpha\}$ is locally finite in X.

By induction we can construct successively systems $\{H_{\alpha i}\}$, $i=2, 3, \dots$ of open F_σ-sets of X such that

$$(a_i) \qquad F_{\alpha i} \smile \overline{H}_{\alpha, i-1} \subset H_{\alpha i}, \quad \overline{H}_{\alpha i} \subset (G_\alpha \smile (X-A)) \frown V_\alpha,$$

$$(b_i) \qquad \{\overline{H}_{\alpha i}\} \text{ is similar to } \{F_{\alpha i} \smile \overline{H}_{\alpha, i-1}\}.$$

Let us put

$$H_\alpha = \overset{\infty}{\underset{i=1}{\smile}} H_{\alpha i}.$$

Then these H_α are open F_σ-sets of X and satisfy (a). It is also obvious that $\{H_\alpha\}$ is locally finite in X. To prove (b), let

$$\overset{r}{\underset{i=1}{\frown}} G_{\alpha_i}=0.$$

Then for any $m \geqq 1$ we have

$$\overset{r}{\underset{i=1}{\frown}} (F_{\alpha_i m} \smile \overline{H}_{\alpha_i, m-1}) = \underset{\Delta}{\smile} (\underset{i\in\Delta}{\frown} F_{\alpha_i m} \frown (\underset{j\bar{\in}\Delta}{\frown} \overline{H}_{\alpha_j, m-1})) \smile (\overset{r}{\underset{i=1}{\frown}} \overline{H}_{\alpha_i, m-1}),$$

where Δ ranges over non-empty subsets of $\{1, 2, \dots, r\}$. Since

$$\underset{i\in\Delta}{\frown} F_{\alpha_i m} \frown (\underset{j\bar{\in}\Delta}{\frown} \overline{H}_{\alpha_j, m-1}) \subset \overset{r}{\underset{i=1}{\frown}} G_{\alpha_i}=0,$$

we see that

$$\overset{r}{\underset{i=1}{\frown}} (F_{\alpha_i m} \smile \overline{H}_{\alpha_i, m-1}) = \overset{r}{\underset{i=1}{\frown}} \overline{H}_{\alpha_i, m-1}.$$

By induction with respect to m we can easily verify by virtue of (b_m) that

$$\overset{r}{\underset{i=1}{\frown}} \overline{H}_{\alpha_i m}=0, \quad \text{for every} \quad m \geqq 1.$$

Thus we have $\bigcap_{i=1}^{r} H_{\lambda_i}=0$. This proves ($b$).

Remark. From the above proof it is seen that if A is a closed set of a paracompact normal space X and $\{G_\alpha\}$ is a locally finite open covering of A, there exists a locally finite open covering $\{H_\alpha\}$ of X satisfying only (a).[4]

Lemma 2. *Let $\{A, B\}$ be a closed covering of a topological space X and $\{U_\alpha\}$ a locally finite system in A which consists of open F_σ-sets U_α of A. If B is paracompact and normal, there exists a locally finite system $\{V_\alpha\}$ of open F_σ-sets of X such that*

(a) $$U_\alpha=V_\alpha\cap A,$$

(b) $\{V_\alpha\}$ *is similar to* $\{U_\alpha\}$.

Proof. If we put $G_\alpha=U_\alpha\cap B$, then $\{G_\alpha\}$ is a locally finite system which consists of open F_σ-sets of $A\cap B$. Applying Lemma 1, we can find a locally finite system $\{H_\alpha\}$ such that H_α are open F_σ-sets of B and $G_\alpha=H_\alpha\cap A\cap B$ for each α and $\{H_\alpha\}$ is similar to $\{G_\alpha\}$. Let us put

$$V_\alpha=U_\alpha\cup H_\alpha.$$

Then these V_α are open F_σ-sets of X since $V_\alpha\cap A=U_\alpha$, $V_\alpha\cap B=H_\alpha$, and satisfy the conditions (a) and (b).

§2. **Proof of Theorem 1.** Since the "only if" part is obvious, we have only to prove the "if" part. Our proof is obtained from an elaboration of the method given in [4].

Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. Let each A_α be paracompact and normal. Then by [4, Theorem 2] X is normal. Hence it is sufficient to prove the paracompactness of X.

Let us assume that the set of indices α consists of all ordinals less than a fixed ordinal η, and put for each $\tau<\eta$

$$P_\tau=\cup\{A_\alpha \mid \alpha\leqq\tau\},\qquad Q_\tau=\cup\{A_\alpha \mid \alpha<\tau\}.$$

Let $\mathfrak{G}$ be any open covering of X. We shall prove the existence of a locally finite refinement $\mathfrak{V}$ of $\mathfrak{G}$. The construction of $\mathfrak{V}$ will be performed by transfinite induction. For this purpose let us assume that for each α less than τ ($<\eta$) there exists a countable collection of locally finite open coverings

$$\mathfrak{U}(\alpha, i)=\{U(\lambda, \alpha, i) \mid \lambda\in\Omega(\alpha, i)\},\ i=0, 1, 2, \dots$$

4) R. Arens has shown that a weaker assertion than this is essential for the validity of a generalization of Tietze's extension theorem. Cf. R. Arens: Extension of coverings, of pseudometrics, and of linear-space-valued mappings, Canadian Jour. Math., **5**, 211–215 (1953).

of P_α with the following properties:

(a_α) $\mathfrak{U}(\alpha, 0)$ is a refinement of $\mathfrak{G} \frown P_\alpha = \{G \frown P_\alpha \mid G \in \mathfrak{G}\}$: $U(\lambda, \alpha, 0) \subset G(\lambda, \alpha) \in \mathfrak{G}$.

(b_α) In case $\beta < \alpha$ we have $\Omega(\beta, i) \subset \Omega(\alpha, i)$, $i = 0, 1, 2, \dots$.

(c_α) In case $\beta < \alpha$ we have $G(\lambda, \beta) = G(\lambda, \alpha)$ for $\lambda \in \Omega(\beta, 0)$.

(d_α) There exists for each $\lambda \in \Omega(\alpha, i)$ a continuous mapping $\varphi_{\lambda,\alpha,i}$ of P_α into a closed unit interval $I = \{t \mid 0 \leqq t \leqq 1\}$ such that

$$U(\lambda, \alpha, i) = \{x \mid \varphi_{\lambda,\alpha,i}(x) > 0\}.$$

(e_α) In case $\beta < \alpha$ we have

$$\varphi_{\lambda,\beta,i} = \varphi_{\lambda,\alpha,i} \mid P_\beta \quad \text{for} \quad \lambda \in \Omega(\beta, i).$$

(f_α) For any $\lambda \in \Omega(\alpha, i)$ the set

$$\Gamma(\lambda, \alpha, i) = \{\mu \mid U(\lambda, \alpha, i) \frown U(\mu, \alpha, i-1) \neq 0,\ \mu \in \Omega(\alpha, i-1)\}$$

is finite, $i = 1, 2, \dots$.

(g_α) In case $\beta < \alpha$ we have

$$\Gamma(\lambda, \beta, i) = \Gamma(\lambda, \alpha, i) \quad \text{for} \quad \lambda \in \Omega(\beta, i).$$

Let us put

$$\Omega_*(i) = \smile \{\Omega(\alpha, i) \mid \alpha < \tau\}, \tag{1}$$

$$\Gamma_*(\lambda, i) = \smile \{\Gamma(\lambda, \alpha, i) \mid \alpha < \tau\}, \tag{2}$$

$$U_*(\lambda, i) = \smile \{U(\lambda, \alpha, i) \mid \alpha < \tau\}, \tag{3}$$

where $\Gamma(\lambda, \alpha, i)$ and $U(\lambda, \alpha, i)$ mean the empty set for $\lambda \in \Omega_*(i) - \Omega(\alpha, i)$.

Then by (g_α) we have $\Gamma_*(\lambda, i) = \Gamma(\lambda, \alpha, i)$ for $\lambda \in \Omega(\alpha, i)$ and $\Gamma_*(\lambda, i)$ is finite. Since by (e_α)

$$U_*(\lambda, i) \frown P_\alpha = U(\lambda, \alpha, i), \tag{4}$$

$U_*(\lambda, i)$ are open sets of Q_τ by the property of weak topology.

For $\lambda \in \Omega_*(i)$, the map $\psi_{\lambda,i} : Q_\tau \to I$ defined by

$$\psi_{\lambda,i}(x) = \varphi_{\lambda,\alpha,i}(x) \quad \text{for} \quad x \in P_\alpha$$

is single-valued and continuous by (e_α) and the property of weak topology of X, and

$$U_*(\lambda, i) = \{x \mid \psi_{\lambda,i}(x) > 0\}. \tag{5}$$

From (2) and (3) it follows that

$$U_*(\lambda, i) \frown U_*(\mu, i-1) = 0 \quad \text{for} \quad \mu \in \Omega_*(i-1) - \Gamma_*(\lambda, i), \tag{6}$$

since in case $\beta \leqq \alpha$, $U(\lambda, \beta, i) \frown U(\mu, \alpha, i-1) = U(\lambda, \alpha, i) \frown U(\mu, \beta, i-1) = 0$.

Therefore $\{U_*(\lambda, i) \mid \lambda \in \Omega_*(i)\}$ $(i = 0, 1, 2, \dots)$ are locally finite open coverings of Q_τ.

Since A_τ is paracompact we can apply Lemma 2 to our case in view of (5); there exists a locally finite system $\{L^{(i)}(\lambda, i),\ L^{(i)}(\mu, i+1) \mid \lambda \in \Omega_*(i),\ \mu \in \Omega_*(i+1)\}$ of open sets of P_τ which is similar to $\{U_*(\lambda, i),\ U_*(\mu, i+1) \mid \lambda \in \Omega_*(i),\ \mu \in \Omega_*(i+1)\}$ and satisfies

$$U_*(\lambda, i)=L^{(i)}(\lambda, i)\frown Q_\tau,\ \ U_*(\mu, i+1)=L^{(i)}(\mu, i+1)\frown Q_\tau.$$

We put

$$H(\lambda, 0)=L^{(0)}(\lambda, 0)\frown G(\lambda),$$
$$H(\lambda, i)=L^{(i-1)}(\lambda, i)\frown L^{(i)}(\lambda, i), \quad i=1, 2, \dots,$$

where $G(\lambda)$ means $G(\lambda, \alpha)$ (with a suitable α) in (a_α) which is determined uniquely by (c_α), and hence $U_*(\lambda, i)\subset G(\lambda) \in \mathfrak{G}$. Here we note that

$$(7) \qquad U_*(\lambda, i)=H(\lambda, i)\frown Q_\tau,$$

$$(8) \qquad \{H(\lambda, i),\ H(\mu, i+1) \mid \lambda, \mu\} \text{ is similar to } \{U_*(\lambda, i),\ U_*(\mu, i+1) \mid \lambda, \mu\}.$$

By the normality of P_τ we can construct a continuous mapping $\varphi_{\lambda,\tau,0} : P_\tau \to I$ such that

$$\varphi_{\lambda,\tau,0}(x)=\begin{cases}\psi_{\lambda,0}(x), & \text{for } x \in Q_\tau,\\ 0, & \text{for } x \in P_\tau - H(\lambda, 0).\end{cases}$$

Let us put

$$U(\lambda, \tau, 0)=\{x \mid \varphi_{\lambda,\tau,0}(x)>0\}.$$

Then $U(\lambda, \tau, 0)\subset H(\lambda, 0)$ and hence $\{U(\lambda, \tau, 0) \mid \lambda \in \Omega_*(0)\}$ is locally finite in P_τ and we have $U(\lambda, \tau, 0)\subset G(\lambda) \in \mathfrak{G}$. By (8) we have

$$(9) \qquad U(\mu, \tau, 0)\frown H(\lambda, 1)=0, \quad \text{for} \quad \mu \in \Omega_*(0)-\Gamma_*(\lambda, 1).$$

Since P_τ is normal there exists an open set N_0 of P_τ such that

$$Q_\tau\subset N_0, \quad \overline{N}_0\subset M_0, \quad M_0=\smile\{U(\lambda, \tau, 0) \mid \lambda \in \Omega_*(0)\}.$$

Since A_τ is paracompact and $P_\tau - M_0=A_\tau - M_0$, there exists a locally finite system $\{U(\nu, \tau, 0) \mid \nu \in \Omega_{**}(0)\}$ in A_τ such that $U(\nu, \tau, 0)$ are open F_σ-sets of A_τ (and hence of P_τ by (10)) and

$$(10) \qquad P_\tau - M_0\subset \smile\{U(\nu, \tau, 0) \mid \nu \in \Omega_{**}(0)\}\subset A_\tau - \overline{N}_0=P_\tau-\overline{N}_0,$$

$$(11) \qquad \{U(\nu, \tau, 0) \mid \nu \in \Omega_{**}(0)\} \text{ is a refinement of } \mathfrak{G}\frown A_\tau.$$

It is obvious that there exists for each $\nu \in \Omega_{**}(0)$ a continuous mapping $\varphi_{\nu,\tau,0} : P_\tau \to I$ such that

$$U(\nu, \tau, 0)=\{x \mid \varphi_{\nu,\tau,0}(x)>0\}.$$

Let us put

$$\mathfrak{U}(\tau, 0)=\{U(\lambda, \tau, 0) \mid \lambda \in \Omega(\tau, 0)\}, \quad \Omega(\tau, 0)=\Omega_*(0) \cup \Omega_{**}(0).$$

Then $\mathfrak{U}(\tau, 0)$ is a locally finite open covering of P_τ and a refinement of $\mathfrak{G} \frown P_\tau$. We shall next construct locally finite open coverings

$$\mathfrak{U}(\tau, i)=\{U(\lambda, \tau, i) \mid \lambda \in \Omega(\tau, i)\}, \quad i=1, 2, \ldots$$

of P_τ satisfying the conditions:

(a_i^*) $\Omega(\tau, i)$ is the sum of two disjoint sets $\Omega_*(i)$ and $\Omega_{**}(i)$.

(b_i^*) There exists for each $\lambda \in \Omega(\tau, i)$ a continuous mapping $\varphi_{\lambda,\tau,i}$: $P_\tau \to I$ such that

$$U(\lambda, \tau, i)=\{x \mid \varphi_{\lambda,\tau,i}(x)>0\}.$$

(c_i^*) For $\lambda \in \Omega_*(i)$ we have

$$\varphi_{\lambda,\tau,i}(x)=\begin{cases} \psi_{\lambda,i}(x), & \text{for } x \in Q_\tau, \\ 0, & \text{for } x \in P_\tau - H(\lambda, i) \frown N_{i-1}, \end{cases}$$

where N_{i-1} is an open set of P_τ such that

$$Q_\tau \subset N_{i-1} \subset \overline{N}_{i-1} \subset \cup \{U(\lambda, \tau, i-1) \mid \lambda \in \Omega_*(i-1)\}.$$

(d_i^*) $A_\tau - \cup\{U(\lambda, \tau, i) \mid \lambda \in \Omega_*(i)\} \subset \cup \{U(\nu, \tau, i) \mid \nu \in \Omega_{**}(i)\} \subset A_\tau - \overline{N}_i$.

(e_i^*) $\Gamma(\lambda, \tau, i)=\{\mu \mid U(\lambda, \tau, i) \frown U(\mu, \tau, i-1) \neq 0, \ \mu \in \Omega(\tau, i-1)\}$ is a finite set and $\Gamma(\lambda, \tau, i)=\Gamma_*(\lambda, i)$ for $\lambda \in \Omega_*(i)$.

If we put $N_{-1}=P_\tau$, then $\mathfrak{U}(\tau, 0)$ defined above satisfies these conditions except (e_i^*) with $i=0$. Let us assume that $\mathfrak{U}(\tau, i-1)$ satisfying conditions (a_{i-1}^*) to (e_{i-1}^*) is constructed. Then $\{U(\lambda, \tau, i) \mid \lambda \in \Omega_*(i)\}$ defined by (b_i^*), (c_i^*) is locally finite in P_τ since $U(\lambda, \tau, i) \subset H(\lambda, i)$ and $\{H(\lambda, i) \mid \lambda\}$ is locally finite in P_τ. Since $\mathfrak{U}(\tau, i-1)$ is locally finite and A_τ is paracompact we can construct a locally finite system $\{U(\nu, \tau, i) \mid \nu \in \Omega_{**}(i)\}$ which is locally finite in A_τ and hence in P_τ and satisfies (b_i^*), (d_i^*) and (e_i^*). The validity of (e_i^*) for $\lambda \in \Omega_*(i)$ now follows from (d_{i-1}^*) and (8) since $U(\lambda, \tau, i) \subset N_{i-1}$ by (c_i^*). Thus the existence of a locally finite open covering $\mathfrak{U}(\tau, i)$ of P_τ satisfying the conditions (a_i^*) to (e_i^*) is verified.

Therefore by induction we see the existence of $\mathfrak{U}(\tau, i)$ satisfying conditions (a_i^*) to (e_i^*) for $i=1, 2, \ldots$.

Then the coverings $\mathfrak{U}(\tau, i)$, $i=0, 1, 2, \ldots$ clearly satisfy the conditions (a_τ) to (g_τ).

Thus by transfinite induction we can find locally finite open coverings $\mathfrak{U}(\alpha, i)$, $i=0, 1, 2, \ldots$ satisfying the conditions (a_α) to (g_α) for any $\alpha<\eta$.

Let us put finally

$$\mathfrak{V}(i)=\{V(\lambda, i) \mid \lambda \in \Omega(i)\},$$

where

$$V(\lambda, i) = \smile \{U(\lambda, \alpha, i) \mid \alpha < \eta\}, \quad \Omega(i) = \smile \{\Omega(\alpha, i) \mid \alpha < \eta\},$$

and $U(\lambda, \alpha, i)$ means the empty set for $\lambda \bar{\in} \Omega(\alpha, i)$. Clearly each $\mathfrak{V}(i)$ is an open covering of X because of the weak topology of X. By the same argument as that for $\{U_*(\lambda, i) \mid \lambda \in \Omega_*(i)\}$ (cf. (6)) we see that each element of $\mathfrak{V}(i)$ intersects only a finite number of sets of $\mathfrak{V}(i-1)$. Hence each $\mathfrak{V}(i)$ is a locally finite covering of X. In particular, $\mathfrak{V}(0)$ is a locally finite open covering of X and a refinement of $\mathfrak{G}$. Thus Theorem 1 is completely proved.

§3. **Some Remarks.** From the above proof of Theorem 1 we have

Corollary. *Let $\{A_i\}$ be a countable closed covering of a topological space X such that a subset of X is closed if its intersection with each A_i is closed.*[5] *If each A_i is normal, then X is normal, and moreover if each A_i is paracompact, then X is paracompact.*

By [1, Theorem 4] and [4, Lemma 3 and Theorem 2] we can easily prove

Theorem 2. *In Theorem 1 the word "paracompact" can be replaced by "countably paracompact" throughout.*

References

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5) For instance, in case the interiors of A_i cover X this condition is satisfied. (But in this case if we put $C_i = A_i - \smile_{j<i} \operatorname{Int} A_j$, $\{C_i\}$ is a locally finite closed covering.)