31. On Blocks of Characters of the Symmetric Group

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(Comm. by K. SHODA, M.J.A., March 12, 1955)

The following basic theorem in the modular representation theory of the symmetric group S_n has been proved in various ways (2, 5).

Theorem. *Two irreducible representations of S_n belong to the same block if and only if they have the same p-core.*

In the present paper we shall give a new proof of this theorem.

1. Let $[\alpha]$ be a Young diagram of n nodes which contains α_i nodes in its ith row and α'_j nodes in its jth column:

$$n=\sum_i \alpha_i=\sum_j \alpha'_j. \tag{1}$$

We denote by χ_α the character of the irreducible representation of S_n associated with $[\alpha]$ and by f_α its degree.

The node in the ith row and jth column of $[\alpha]$ is called its ij-node. It is called the corner of the ij-right hook that consists of this node and all nodes to the right of it or below it. Let us denote by $h_{i,j}$ the total hook length of the ij-right hook. The hook product H_α of $[\alpha]$ is the product of the n integers $h_{i,j}$ (3). Then we have

$$f_\alpha=n!/H_\alpha. \tag{2}$$

Lemma 1. *If the kl-right hook of length $h_{k,l}=g$ is removed from $[\alpha]$ leaving $[\gamma]$, then*

$$f_\alpha/f_\gamma=\frac{n!}{(n-g)!\,g!}KLM \text{ with } K=\prod_{i<k}((h_{i,l}-g)/h_{i,l}),$$
$$L=\prod_{j<l}((h_{k,j}-g)/h_{k,j}), \qquad M=\prod_{k<i\leqq\alpha'_l}((g-h_{i,l})/h_{i,l}).$$

Proof. We denote by $h'_{i,j}$ the total hook length of the ij-right hook of $[\gamma]$. We see easily that

$$h'_{i,j}=\begin{cases} h_{i+1,j} & \text{if } k\leqq i<\alpha'_l,\ j<l, \\ h_{k,j}-g & \text{if } i=\alpha'_l,\ j<l, \\ h_{i,j+1} & \text{if } i<k,\ l\leqq j<\alpha_k, \\ h_{i,l}-g & \text{if } i<k,\ j=\alpha_k, \\ h_{i+1,j+1} & \text{if } k\leqq i,\ l\leqq j, \\ h_{i,j} & \text{otherwise.} \end{cases} \tag{3}$$

Moreover we have (3, Lemma 1)

$$\prod_{l\leqq j\leqq\alpha_k} h_{k,j} \prod_{k<i\leqq\alpha'_l}(g-h_{i,l})=g!. \tag{4}$$

The lemma is proved easily by (2)–(4).

If we set $\beta_i=h_{i,1}$, $\beta'_j=h_{1,j}$, then we see that Lemma 1 is identical with the lemma (4, p. 101) since

$$h_{i,l}-g=\beta_i-\beta_k \ (i<k),\ h_{k,j}-g=\beta'_j-\beta'_l \ (j<l),$$
$$g-h_{i,l}=\beta_k-\beta_i \ (k<i\leqq \alpha'_l).$$

In what follows we set $g=p$, a prime number. Let $[\alpha]$ be a diagram of n nodes with p-core $[\alpha_0]$ and $[\alpha]^*$ be its star diagram. Suppose that $[\alpha]$ is of weight b. We then have by (3)

$$(5) \qquad H_\alpha=p^b H_{\alpha^*} H'_\alpha,$$

where H'_α is the product of all $h_{i,j}$ which are prime to p and H_{α^*} denotes the hook product of $[\alpha]^*$.

Lemma 2. *If the kl-right hook of length $h_{k,l}=p$ is removed from $[\alpha]$ leaving $[\gamma]$, then*

$$H'_\alpha \equiv (-1)^{r+1} H'_\gamma \qquad (\mathrm{mod}\ p),$$

where r denotes the leg length of the kl-right hook.

Proof. It follows from (4) that

$$\prod_{l<j\leqq \alpha_k} h_{k,j} \prod_{k<i\leqq \alpha'_l} (p-h_{i,l})=(p-1)!\equiv -1 \qquad (\mathrm{mod}\ p),$$

whence

$$\prod_{l<j\leqq \alpha_k} h_{k,j} \prod_{k<i\leqq \alpha'_l} h_{i,l}\equiv (-1)^{r+1} \qquad (\mathrm{mod}\ p).$$

This, combined with (3), yields our assertion.

Using Lemma 2 we obtain by induction the

Lemma 3. *If the p-core $[\alpha_0]$ is obtained from $[\alpha]$ by removing successively b p-hooks T_i of leg length r_i, then*

$$(6) \qquad H'_\alpha \equiv (-1)^{\sigma+b} H_{\alpha_0} \qquad (\mathrm{mod}\ p),$$

where $\sigma=\sum_i r_i$.

If we denote by χ_{α^*} the character of the reducible representation $[\alpha]^*$ of S_b associated with the star diagram $[\alpha]^*$ and by f_{α^*} its degree, then we have by (3)

$$(7) \qquad f_{\alpha^*}=b!/H_{\alpha^*}.$$

We shall set $\omega_\alpha(G)=g(G)\chi_\alpha(G)/f_\alpha$, where $g(G)$ is the number of elements in the class of G. Let G be an element possessing b p-cycles and let G_0 be the element of S_{n-bp} obtained from G by removing those b p-cycles. Then we obtain by (9)

$$(8) \qquad \chi_\alpha(G)=(-1)^\sigma f_{\alpha^*}\chi_{\alpha_0}(G_0).$$

Suppose that G has exactly b p-cycles. Using (2), (5)–(8) we have the relation (11) in (5):

$$(9) \qquad \omega_\alpha(G)\equiv(-1)^b\omega_{\alpha_0}(G_0) \qquad (\mathrm{mod}\ p).$$

(Observe that $\omega_\alpha(G)$ and $\omega_{\alpha_0}(G_0)$ are rational integers and H_{α_0} is prime to p.) This congruence (9) holds however also for those elements G which possess more than b p-cycles; for the both sides vanish then. Thus we obtain as in (5):

If two irreducible representations of S_n belong to the same block, then they have the same p-core.

Remark. Applying the Murnaghan-Nakayama recursion formula we have $\sum_\alpha \chi_\alpha(V)\chi_\alpha(S)=0$ for any p-regular V and for any p-singular

S, where the sum extends over all $[\alpha]$ of S_n with the same p-core. Hence we can derive also the same result (8, Theorem 3).

We set $n=n'+ap$, where $0\leqq n'<p$. Denote by $t(l)$ the number of p-cores with $n'+lp$ nodes. We have by the above discussion

$$\sum_{l=0}^{a} t(l) \leqq s(n), \tag{10}$$

where $s(n)$ denotes the number of blocks of S_n. In section 2 we shall prove that the equality sign holds in (10).

2. We shall apply the general theory of blocks of characters (1, §§1–4) to S_n. Let $\mathfrak{H}$ be any p-subgroup of S_n and let its order be p^h, $h>0$. We consider a subgroup $\mathfrak{N}$ which satisfies the condition

$$\mathfrak{H}C(\mathfrak{H})\subseteq\mathfrak{N}\subseteq N(\mathfrak{H}). \tag{11}$$

Denote the center of the modular group ring $\Gamma^*(S_n)$ by Λ^*. As was shown in (1), there exists the ideal T^* such that

$$R^* \cong \Lambda^* / T^*, \tag{12}$$

where R^* denotes the subring of the center $\Lambda^*(\mathfrak{N})$ of the modular group ring $\Gamma^*(\mathfrak{N})$.

We consider a block B of weight b with the defect group $\mathfrak{D}$. The defect d of B is zero if and only if $b=0$. Now we assume that $b>0$. Then $\mathfrak{D}$ contains an element $Q=P_1.P_2 \dots P_m$ of order p, where no two of P_i have common symbols and each P_i is a p-cycle. We have

$$N(Q)\cong S_{n-mp}\times S(m, p), \tag{13}$$

where $S(m, p)$ is the generalized symmetric group (6, 7) and consists of those permutations which transform the cycles P_i into each other. Let $\mathfrak{P}_m$ be the p-Sylow-subgroup of $S(m, p)$. Since $S(m, p)$ possesses only one block (for p), the defect group of every block of $N(Q)$ contains $\mathfrak{P}_m$ (2, §2, IX). In (11) we now take $\mathfrak{H}$ as the group generated by Q and $\mathfrak{N}=N(Q)$. Let E^* be the primitive idempotent element of Λ^* that corresponds to B. Then E^* does not lie in T^* in (12) since $Q \in \mathfrak{D}$ (8). Consequently we have $\mathfrak{P}_m\subseteq\mathfrak{D}$ (1, Theorem 1). Thus we have proved that the defect group of every block of a positive defect contains a p-cycle. Hence we may assume without restriction that every defect group $\neq 1$ contains a fixed p-cycle P. Now we take $\mathfrak{H}$ in (11) as the group generated by P and $\mathfrak{N}=N(P)=S_{n-p}\times\{P\}$. By our assumption every primitive idempotent element of Λ^* that corresponds to a block of a positive defect does not lie in T^*. It follows from (12) that

$$s(n)-t(a)\leqq s(n-p), \tag{14}$$

since $N(P)$ possesses $s(n-p)$ blocks and R^* is the subring of the center $\Lambda^*(N(P))$. Now we shall prove our theorem by induction. Since $t(0)=s(n')$, we shall assume that $\sum_{l=0}^{k} t(l)=s(n'+kp)$ for $k<a$.

We then obtain by (14)

$$s(n) \leqq t(a) + \sum_{l=0}^{a-1} t(l) = \sum_{l=0}^{a} t(l). \tag{15}$$

(10) and (15) yield $s(n) = \sum_{l=0}^{a} t(l)$, and the proof of our theorem is complete.

3. Let B be a block of weight b with p-core $[\alpha_0]$. We shall determine the defect group of B. If $e(a)$ denotes the exponent of the highest power of p dividing an integer a, then

$$e(n!/f_\alpha) = (e(bp)!) - e(f_{\alpha^*}), \qquad [\alpha] \subset B. \tag{16}$$

Since $(f_{\alpha^*}, p) = 1$ for a suitable $[\alpha] \subset B$, the defect $d(b)$ of B is given by

$$e((bp)!) = b + e(b!). \tag{17}$$

We consider an element $Q_b = P_1.P_2 \ldots P_b$ of order p. Then $N(Q_b) = S_{n-bp} \times S(b, p)$. Let $\mathfrak{P}_b$ be the p-Sylow-subgroup of $S(b, p)$. The order of $\mathfrak{P}_b$ is $d(b)$. Since $[\alpha_0]$ is the p-core, if we choose a suitable p-regular element G_0 of S_{n-bp} such that the order of the normalizer $N^*(G_0)$ of G_0 in S_{n-bp} is prime to p, then $\omega_{\alpha_0}(G_0) \not\equiv 0 \pmod{p}$. Let G' be an element of S_n possessing bp 1-cycles and assume that G_0 is obtained from G' by removing those bp 1-cycles. Then $\mathfrak{P}_b$ is the p-Sylow-subgroup of the normalizer $N(G')$ of G' in S_n. Now we choose $\mathfrak{H}$ in (11) as the group generated by Q_b and $\mathfrak{N} = N(Q_b)$. We then have by the relation (5) in (1) the following congruence (see 5, p. 117):

$$\omega_\alpha(G') \equiv \omega_{\alpha_0}(G_0) \not\equiv 0 \qquad \pmod{p}, \tag{18}$$

whence $\mathfrak{D} \subseteq \mathfrak{P}_b$ (8). Since two groups have the same order, we obtain

$$\mathfrak{D} = \mathfrak{P}_b. \tag{19}$$

This proves Theorem 1 (2).

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