101. On the Group of Conformal Transformations of a Riemannian Manifold

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(Comm. by K. KUNUGI, M.J.A., July 12, 1955)

Let M be a connected and differentiable Riemannian manifold of dimension $n(\geqq 3)$ with the fundamental metric tensor field G. A differentiable homeomorphism φ of M onto M is called a *conformal transformation* if $(\varphi G)_p=\rho(p)G_p$ at every point p of M where ρ is a positive function on M determined by φ and called the *associated function* of φ. If in particular the function ρ is constant, φ is called a *homothetic transformation*. And if furthermore $\rho=1$ at every point of M, φ is said to be an *isometric transformation* or *isometry*.

We denote by $K(M)$, $H(M)$, and $I(M)$ the group of all conformal transformations, that of all homothetic ones and that of all isometries of M respectively. It is then clear that we have $K(M)\supset H(M)\supset I(M)$. As is well known a conformal transformation leaves invariant the Weyl's conformal curvature tensor field C of M.

Now we denote by $K_p(M)$ the group of isotropy of $K(M)$ at a point p of M. If $\varphi\in K_p(M)$, φ induces a linear transformation $\tilde{\varphi}$ on the tangent vector space T_p at p. This correspondence $\varphi\to\tilde{\varphi}$ is a linear representation[1] of $K_p(M)$ onto $\tilde{K}_p(M)$ which is a subgroup of the homothetic group $H(n)$ of T_p. If in particular $\tilde{K}_p(M)$ is contained in the orthogonal group $O(n)$ of T_p, p is called to be an *isometric point*. If $\tilde{K}_p(M)$ is not contained in $O(n)$, p is said a *homothetic point*.

We shall first establish

THEOREM 1. *The conformal curvature tensor field C of M vanishes at any homothetic point.*

This theorem will be obtained as a corollary to the following lemma.

LEMMA. *Let V be an n-dimensional vector space over the real number field and $\tilde{\varphi}$ a homothetic transformation which is not an orthogonal one. If a tensor S of type (p, q), $p\neq q$, is invariant by $\tilde{\varphi}$, then S is the zero tensor.*

PROOF. We regard S as a multilinear mapping of

$$\underbrace{V\times\cdots\times V}_{p}\times\underbrace{V^*\times\cdots\times V^*}_{q}$$

1) In general this linear representation is not faithful.

into the real number field such that

$$S(X_1,\dots,X_p,\ \omega_1,\dots,\omega_q)=S(\tilde{\varphi}X_1,\dots,\tilde{\varphi}X_p,\ \omega_1\tilde{\varphi},\dots,\tilde{\varphi}\omega_q)^{2)}$$

for any $X_1,\dots,X_p$ in V and $\omega_1,\dots,\omega_q$ in V^*, where V^* is the dual space of V. S may be extended, in a unique manner, to a multilinear mapping S^K of

$$\underbrace{V^K\times\cdots\times V^K}_{p}\ \times\ \underbrace{V^{*K}\times\cdots\times V^{*K}}_{q}$$

into the complex number field K, where V^K is the vector space over K deduced from V by extending to K the basic field. This extension S^K is invariant by $\tilde{\varphi}$ on V^K. Since $\tilde{\varphi}$ is a homothetic transformation, it may be written in one and only one way in the form $\tilde{\varphi}=e^{\rho}\varepsilon\cdot\sigma$, where ρ is a real number $\neq 0$, ε the identity transformation and σ a real orthogonal transformation. Being orthogonal, σ is representable as a matrix of the form

$$\begin{pmatrix} e^{\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{\theta_n} \end{pmatrix}$$

with respect to a certain base $\{X_1,\dots,X_n\}$ of V^K, where $\theta_1,\dots,\theta_n$ are 0 or pure imaginary numbers. Then $\tilde{\varphi}$ has the form

$$\begin{pmatrix} e^{\rho+\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{\rho+\theta_n} \end{pmatrix}$$

i.e. $\tilde{\varphi}X_i=e^{\rho+\theta_i}X_i$, and $\tilde{\varphi}\omega_i=e^{-(\rho+\theta_i)}\omega_i$ $(1\leqq i\leqq n)$, where $\{\omega_1,\dots,\omega_n\}$ is the base of V^* defined by $\omega_i(X_j)=\delta_{ij}$. Let $\{i_1,\dots,i_p\}$ and $\{j_1,\dots,j_q\}$ be arbitrary sequences of indices consisting of $1,\dots,n$, then, S^K being invariant by $\tilde{\varphi}$, we have

$$\begin{aligned} &S^K(X_{i_1},\dots,X_{i_p},\ \omega_{j_1},\dots,\omega_{j_q})\\ &=S^K(\tilde{\varphi}X_{i_1},\dots,\tilde{\varphi}X_{i_p},\ \tilde{\varphi}\omega_{j_1},\dots,\tilde{\varphi}\omega_{j_q})\\ &=e^{A}S^K(X_{i_1},\dots,X_{i_p},\ \omega_{j_1},\dots,\omega_{j_q}) \end{aligned}$$

where $A=(p-q)\rho+\sum_{\alpha=1}^{p}\theta_{i_\alpha}-\sum_{\beta=1}^{q}\theta_{j_\beta}$. From the assumption $p\neq q$, we have $e^A\neq 1$ whatever θ_{i_α} and θ_{j_β} may be, and therefore

$$S^K(X_{i_1},\dots,X_{i_p},\ \omega_{j_1},\dots,\omega_{j_q})=0.$$

Since this must hold for all choices of $\{i_1,\dots,i_p\}$ and $\{j_1,\dots,j_q\}$ we can conclude that S^K is the zero tensor, so also is S.

Theorem 1 follows immediately from this lemma, since the conformal curvature tensor C_p at p is of type (3,1) and invariant by the transformations of $\tilde{K}_p(M)$.

As is well known, a three-dimensional Riemannian manifold M is conformally flat if and only if

$$L_{ijk}=0$$

in a coordinate neighbourhood of any point of M, where

2) The notation $\tilde{\varphi}\omega$ means ${}^t\tilde{\varphi}^{-1}\omega$ in the usual notation.

$$L_{ijk}=R_{ij;k}-\frac{1}{4}g_{ij}R_{;k}-R_{ik;j}+\frac{1}{4}g_{ik}R_{;j},$$

in which R_{ij} and R are the Ricci tensor and scalar curvature of M respectively. Since the tensor field L defined above is of type (3,0) and invariant by the conformal transformation, it follows from the foregoing lemma that L vanishes at any homothetic point.

We can now give a sufficient condition for M to be comformally flat.

THEOREM 2. *If $K(M)$ is transitive and there exists a homothetic point in M, then M is conformally flat.*

We shall next consider the conformal transformation from another standpoint of view. In a previous paper[3] we have proved the following

THEOREM. *Let M be a connected Riemannian manifold which is not locally flat and φ a homothetic transformation which is not an isometry. Then φ has no fixed point.*

This can be extended, by a slight modification, to the case in which φ is a conformal transformation.

THEOREM 3. *Let M be a connected Riemannian manifold which is not conformally flat, φ a conformal transformation and ρ the associated function of φ, i.e. $\varphi(G)_p=\rho(p)G_p$, $p\in M$. If ρ satisfies the following inequality at every point p of M*

$$\rho(p)<1-\varepsilon \quad \text{or} \quad \rho(p)>1+\varepsilon,$$

where ε is a positive constant, then φ has no fixed point.

THEOREM 4. *Let M be a complete and connected Riemannian manifold which is not conformally flat. Then the associated function of any conformal transformation can take the value unity or arbitrary values near to unity.*

REMARK. Let M be a product manifold $L\times N$ of the straight line L and a Riemannian manifold N which is not conformally flat. We give M the Riemannian metric

$$ds^2=e^{(1+f(y)^2)}\ (dx^2+d\sigma^2)$$

where x is the usual coordinate on L, $d\sigma^2$ the Riemannian metric of N and $f(y)$, $y\in N$, a differentiable function of N which is not constant. It is easy to see that the Riemannian manifold M is neither conformally flat *nor complete* with respect to this metric.

Now, for any $x\in L$, $y\in N$ we denote by φ_a the mapping $(x, y)\to(x-a, y)$ of M onto M, where a is a real number. Then φ_a is a conformal transformation and the associated function ρ_a of φ_a is given by $\rho_a(x, y)=e^{a(1+f(y)^2)}$. If $a\neq 0$, φ_a has no fixed point and ρ_a satisfies

3) Cf. S. Ishihara-M. Obata: Affine transformations on a Riemannian manifold, Tôhoku Math. J., to appear.

$$\rho_a(x, y) < 1-\varepsilon \quad \text{or} \quad \rho_a(x, y) > 1+\varepsilon$$

for some positive number ε.

If furthermore M is compact, orientable and connected Riemannian manifold, we can consider the total volume of M, which is, as a matter of course, invariant by any conformal transformation. From this we have

THEOREM 5. *On a compact, orientable and connected Riemannian manifold the associated function of any conformal transformation takes the value unity.*