165. On Coverings and Continuous Functions of Topological Spaces

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The purpose of this paper is to study relations between continuous functions and locally finite coverings playing the important rôle in recent topological developments. We shall establish a necessary and sufficient condition for a normal space to be fully normal and a condition for metrizability by using families of continuous functions and shall generalize Hausdorff's extension theorem of continuous function by using coverings.

Lemma. *Let R be a topological space and $V_\alpha=\{x\mid f_\alpha(x)>0\}$*[1] *$(\alpha<\tau)$, where $f_\alpha(\alpha-\tau)$ are real valued functions on R. If $\mathfrak{V}=\{V_\alpha\mid \alpha<\tau\}$ is a covering of R, and if $\bigcup_{\beta<\alpha} f_\alpha(x)$ is continuous for every $\alpha<\tau$, then $\mathfrak{V}$ has a locally finite refinement.*

Proof. Let $V_{1\alpha}=\left\{x\mid f_\alpha(x)>\frac{1}{2}\right\}$ and $V_{i\alpha}=\left\{x\mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}\right\}$ $(n\geqq 2)$, then $\overline{V}_{i\alpha}\subseteq V_{i+1\alpha}\subseteq V_\alpha$ $(i=1,2\ldots)$.

Define $N_{n1}=V_{n1}$, $N_{n\alpha}=V_{n\alpha}-\overline{\bigcup_{\beta<\alpha} V_{n+1\beta}}(1<\alpha<\tau)$, then $\bigcup\{N_{n\alpha}\mid n=1,2,\ldots,\alpha<\tau\}=R$. For $x\in V_1$ implies $x\in V_{n1}=N_{n1}$ for some n, and $x\in V_\alpha$, $x\notin V_\beta(\beta<\alpha)$, $1<\alpha<\tau$ imply $x\in V_{n\alpha}$ for some n and $\bigcup_{\beta<\alpha} f_\beta(x)\leqq 0$. Since $\bigcup_{\beta<\alpha} f_\beta$ is continuous, there exists a nbd (=neighbourhood) $U(x)$ of x such that $U(x)\frown(\bigcup_{\beta<\alpha} V_{n+1\beta})=\phi$. Hence $x\notin\overline{\bigcup_{\beta<\alpha} V_{n+1\beta}}$, and hence $x\in N_{n\alpha}$.

Next, we shall show $\{N_{n\alpha}\mid\alpha<\tau\}$ is locally finite. Let $V'_\alpha=\left\{x\mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{2}\frac{1}{2^{n+1}}\right\}$, then $V'_\alpha\subseteq V_{n+1\alpha}$. If $x\in V'_\alpha$, $x\notin V'_\beta$ $(\beta<\alpha\leqq\tau)$, then $\bigcup_{\beta<\alpha} f_\beta(x)\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2}\frac{1}{2^{n+1}}$. Since $\bigcup_{\beta<\alpha} f_\beta$ is continuous, there exists a nbd $V(x)$ of x such that $V(x)\frown V_{n\beta}=\phi(\beta<\alpha)$. Moreover, $x\in V_{n+1\alpha}$ and $V_{n+1\alpha}\frown N_{n\alpha'}=\phi(\alpha'>\alpha)$. Hence there exists a nbd of x intersecting at most one of $N_{n\alpha}(\alpha<\tau)$. Therefore, $F_n=\bigcup_\alpha \overline{N}_{n\alpha}$ is closed.

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1) α, β, τ denote ordinals in this lemma. In this note covering and refinement mean open covering and open refinement respectively, and notations and terminologies are chiefly due to J. W. Tukey: Convergence and uniformity in topology (1940). The details of the content of this paper will be published in an another place.

Define $V_{n\alpha}=\left\{x \mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{3\cdot 2^{n+1}}\right\}$,

$$V''_{n\alpha}=\left\{x \mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{2}{3\cdot 2^{n+1}}\right\} \text{ and}$$

$$M_{n1}=V'_{n1},\ M_{n\alpha}=V'_{n\alpha}-\overline{\smile_{\beta<\alpha} V''_{n\beta}}\,(1<\alpha<\tau), \text{ then}$$

$\overline{N}_{n\alpha}\subseteq M_{n\alpha}$. $\overline{N}_{n1}\subseteq M_{n1}$ is obvious. If $\alpha\geqq 2$, $x\notin M_{n\alpha}$, then since $\overline{N}_{n\alpha}\subseteq \overline{V}_{n\alpha}\subseteq V'_{n\alpha}$, $x\notin V'_{n\alpha}$ implies $x\notin \overline{N}_{n\alpha}$. Since $x\notin \smile_{\beta<\alpha} V_{n+1\beta}$ implies $\smile_{\beta<\alpha} f_\beta(x) \leqq \frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2^{n+1}}$ and accordingly $\smile_{\beta<\alpha} f_\beta(U(x))\leqq \frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{3\cdot 2^{n+1}}$[2] i.e. $U(x)\frown(\smile_{\beta<\alpha} V''_{n\beta})=\phi$ for some nbd $U(x)$ of x, $x\notin \overline{\smile_{\beta<\alpha} V''_{n\beta}}$. Hence $x\in \overline{\smile_{\beta>\alpha} V''_{n\beta}}$ implies $x\in \smile_{\beta<\alpha} V_{n+1\beta}\subseteq N^c_{n\alpha}$[3] and $x\notin \overline{N}_{n\alpha}$. Thus we get $\overline{N}_{n\alpha}\subseteq M_{n\alpha}$.

Now we denote $W_{1\alpha}=M_{1\alpha}$, $W_{n\alpha}=M_{n\alpha}-\smile_{i=1}^{n-1} F_i\,(n\geqq 2)$.

Then $\mathfrak{W}=\{W_{n\alpha}\mid n=1,2,\dots;\alpha<\tau\}$ is a locally finite refinement of $\mathfrak{V}$. Firstly, we prove $\smile\{W_{n\alpha}\mid n=1,2,\dots;\alpha<\tau\}=R$. Since $\smile \overline{N}_{n\alpha}=R$, for every $x\in R$ there exists n such that $x\in \overline{N}_{n\alpha}$ for some $\alpha<\tau$ and $x\notin \overline{N}_{m\beta}(m<n,\beta<\tau)$. From $\overline{N}_{n\alpha}\subseteq M_{n\alpha}$ we get $x\in M_{n\alpha}$ and $x\notin \smile_{i=1}^{n-1} F_i$, and hence $x\in W_{n\alpha}$.

Since $\mathfrak{W}<\mathfrak{V}$ is obvious, we show lastly that $\mathfrak{W}$ is locally finite. If $x\in N_{k\alpha}\subseteq F_k$, then $N_{k\alpha}\frown W_{m\beta}=\phi\,(m>k,\ \beta<\tau)$. Then we denote $V'_\alpha=\left\{x\mid f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}\right\}$ for $n\leqq k$ and $\alpha<\tau$. If $x\in V'_\gamma$ and $x\notin V'_\beta(\beta<\gamma\leqq\tau)$, then since $\smile_{\beta<\gamma} f_\beta(x)\leqq \frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}$, there exists a nbd $V(x)$ of x such that $V(x)\frown V'_{n\beta}=\phi\,(\beta<\gamma)$. Hence $V(x)\frown M_{n\beta}=\phi$ and $V(x)\frown W_{n\beta}=\phi\,(\beta<\gamma)$. Moreover, $x\in V''_{n\gamma}$ and $V''_{n\gamma}\frown M_{n\alpha'}=\phi\,(\alpha'>\gamma)$. Therefore there exists a nbd $V_n(x)$ of x intersecting at most one of $M_{n\alpha}$ for $n\leqq k$. Hence the nbd $\frown_{i=1}^{k} V_i(x)\frown N_{k\alpha}$ of x intersects only finitely many $W_{n\alpha}$.

From this lemma combining the theorem of A. H. Stone[4] we get easily the following theorems.

Theorem 1. *In order that a T_2-space R is fully normal it is necessary and sufficient that for every open covering $\{V_\alpha\mid\alpha\in A\}$, there exists a family $\{f_\alpha\mid\alpha\in A\}$ of real valued functions on R such that $f_\alpha(V^c_\alpha)=0$, $\smile_{\alpha\in A} f_\alpha=1$, $\smile_{\alpha\in B} f_\alpha$ is continuous for every $B\subseteq A$.*

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2) $f(U)\leqq k$ means $f(x)\leqq k$ $(x\in U)$.

3) We denote by N^C or $C(N)$ the complement of N.

4) A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54** (1948).

Theorem 2. *In order that a completely regular space R is fully normal it is necessary and sufficient that if $\smile_{\alpha\in A}\varphi_\alpha$ is a continuous function on R, then for every $\varepsilon>0$ there exists $\{f_\alpha \mid \alpha\in A\}$ such that $f_\alpha\leqq\varphi_\alpha(\alpha\in A)$, $|\smile_{\alpha\in A}\varphi_\alpha-\smile_{\alpha\in A}f_\alpha|<\varepsilon$, and $\smile_{\beta\in B}f_\beta$ is continuous for every $B\subseteq A$.*

By using this theorem we get the following proposition due to K. Morita.[5]

Corollary 1. *Let R be a normal space and $R=\smile_{n=1}^{\infty}F_n^\circ$.*[7] *If F_n $(n=1,2,\dots)$ are closed and fully normal subspaces, then R is fully normal.*

The following proposition due to K. Nagami[6] is a direct consequence of the above lemma.

Corollary 2. *Let R be a topological space and $V_n=\{x\mid f_n(x)>0\}$ $(n=1,2,\dots)$, where f_n $(n=1,2,\dots)$ are real valued continuous functions on R. If $\mathfrak{V}=\{V_n\mid n=1,2,\dots\}$ is a covering of R, then $\mathfrak{V}$ has a locally finite refinement.*

Theorem 3. *In order that a T_1-space R is metrizable it is necessary and sufficient that there exists a family $\{f_\alpha\mid\alpha\in A\}$ of real valued continuous functions on R such that $\smile_{\beta\in B}f_\beta$ and $\frown_{\beta\in B}f_\beta$ are continuous for every $B\subseteq A$, and such that for every nbd $U(x)$ of x there exists $f_\alpha\in\{f_\alpha\mid\alpha\in A\}$: $f_\alpha(x)<\varepsilon$ and $f_\alpha(U^c(x))\geqq\varepsilon$ for some $\varepsilon>0$.*

Proof. We shall prove the sufficiency. Let $V_{\alpha r}=\{y\mid f_\alpha(y)<r\}$, $W_{\alpha r}=\{y\mid f_\alpha(y)>r\}$ and let $U_{rr'}(B)=(\frown_{\alpha\in B}V_{\alpha r'}\frown_{\alpha\in C(B)}W_{\alpha r})^\circ$ for $B\subseteq A$ and for rational numbers $r'>r>0$, where we define $\frown_{\alpha\in B}V_{\alpha r'}=R$ for $B=\phi$ and $\frown_{\alpha\in C(B)}W_{\alpha r}=R$ for $C(B)=\phi$. Moreover, we define $\mathfrak{U}_{rr'}=\{U_{rr'}(B)\mid B\subseteq A\}$. Putting $A(x)=\left\{\alpha\mid f_\alpha(x)<\frac{r+r'}{2}\right\}$ for a definite $x\in R$, we get $\smile_{\alpha\in A(x)}f_\alpha(x)\leqq\frac{r+r'}{2}$ and consequently $M(x)=\{y\mid\smile_{\alpha\in A(x)}f_\alpha(y)<r'\}\subseteq\frown_{\alpha\in A(x)}V_{\alpha r'}$, $N(x)=\{y\mid\frown_{\alpha\in C(A(x))}f_\alpha(y)>r\}\subseteq\frown_{\alpha\in C(A(x))}W_{\alpha r}$, where $M(x)=R$ for $A(x)=\phi$, $N(x)=R$ for $C(A(x))=\phi$. Since $\smile_\alpha f_\alpha(y)$ and $\frown_\alpha f_\alpha(y)$ are continuous, $M(x)$ and $N(x)$ are open nbd of x such that $M(x)\frown N(x)\subseteq U_{rr'}(A(x))$. Hence $\{M(x)\frown N(x)\mid x\in R\}=\mathfrak{N}<\mathfrak{U}_{rr'}$.

Now we shall show that $\mathfrak{N}$ has a locally finite refinement. Obviously $\smile_\alpha f_\alpha(x)<r'$, if and only if $\frown_\alpha(r+r'-f_\alpha(f_x))<r$. Therefore, $M(x)\frown N(x)$

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5) K. Morita: On spaces having the weak topology with respect to closed coverings. II, Proc. Japan Acad., **30** (1954).

6) K. Nagami: Baire sets, Borel sets and some typical semi-continuous functions, Nagoya Math. Journ., **7** (1954).

7) F_n^0 denotes the interior of F_n.

$=\{y \mid \bigcap_{\alpha\in C(A(x))} f_\alpha(y) \frown \bigcap_{\alpha\in A(x)} (r+r'-f_\alpha(y))>r\}$. To prove the continuity of $\bigcup\{\bigcap_{\alpha\in C(B)} f_\alpha(y) \frown \bigcap_{\alpha\in B}(r+r'-f_\alpha(y)) \mid B\in\mathfrak{B}\}=F(y)$ for an arbitrary $\mathfrak{B}\subseteq 2^A$ we denote by a the value of this function at a definite point y of R. For an arbitrary $\varepsilon>0$ there exists $\alpha\in C(B): f_\alpha(y)<a+\frac{\varepsilon}{2}$ or $\alpha\in B$: $r+r'-f_\alpha(y)<a+\frac{\varepsilon}{2}$ for every $B\in\mathfrak{B}$. We denote by B' the totality of α such that $\alpha\in C(B)$, $f_\alpha(y)<a+\frac{\varepsilon}{2}$ and by B'' the totality of α such that $\alpha\in B$, $r+r'-f_\alpha(y)<a+\frac{\varepsilon}{2}$. Since $\bigcup_{\alpha\in B'} f_\alpha$ is continuous, there exists a nbd $U(y)$ of y such that $\bigcup_{\alpha\in B'} f_\alpha(U(y))<a+\varepsilon$. Since $\bigcup_{\alpha\in B''}(r+r'-f_\alpha(y)) =r+r'-\bigcap_{\alpha\in B''} f_\alpha(y)$ is continuous, there exists a nbd $V(y)$ of y such that $\bigcup_{\alpha\in B''}(r+r'-f_\alpha(V(y)))<a+\varepsilon$. Hence $F(U(y)\frown V(y))\leqq a+\varepsilon$. Since $\bigcap_{\alpha\in C(B)} f_\alpha(y) \frown \bigcap_{\alpha\in B}(r+r'-f_\alpha(y))$ is obviously continuous, there exists a nbd $W(y)$ of y such that $F(W(y))>a-\varepsilon$. Therefore from the above lemma $\mathfrak{N}$ has a locally finite refinement.

Lastly, let $U(x)$ be a nbd of x, then there exists a positive rational number r' such that $x\in V_{\alpha r'}\subseteq U(x)$. Taking a rational number $r>0: f_\alpha(x)<r<r'$, we get $S(x, \mathfrak{U}_{rr'})\subseteq U(x)$. For if $x\in U_{rr'}(B)$, then since $f_\alpha(x)<r$ and consequently $x\notin W_{\alpha r}$, it must be $\alpha\in B$. Hence $U_{rr'}(B)\subseteq V_{\alpha r'}\subseteq U(x)$, and hence $S(x, U_{rr'})\subseteq U(x)$.

Since $\{\mathfrak{U}_{rr'} \mid r, r'$ are rational positive numbers$\}$ is enumerable, we get the metrizability of R from the theorem due to Y. Smirnov and the author.[8)]

Conversely if R is metrizable, then $\{\rho(x,y)\mid x\in R\}$ satisfies the condition of this theorem, where $\rho(x,y)$ denotes a bounded distance of R.

Theorem 4. *Let R be a fully normal uniform space with the uniform topology defined by the uniform coverings $\{\mathfrak{M}_{\alpha'}\mid \alpha'\in A'\}$ and S a uniform space with the uniform topology defined by the uniform coverings $\{\mathfrak{N}_\alpha \mid \alpha\in A\}$ such that $|A'|=|A|=\mathfrak{m}$. If f is a continuous mapping defined on a closed set F of R and having values in S, then S can be imbedded in a uniform space T having a uniform covering system with the cardinal $\mathfrak{m}$ such that f can be continuously extended to R with values in T such that the extension is a homeomorphism of $R-F$ with $T-S$, and such that S is a closed sub-uniform space of T. If f is a homeomorphism, then the extension is also a homeomorphism.*

8) Y. Smirnov: A necessary and sufficient condition for metrizability of topological space, Doklady Akad. Nauk SSSR. N. S., **77** (1951). J. Nagata: On a necessary and sufficient condition of metrizability, Journ. Inst. Polytech. Osaka City Univ., **1**, No. 2 (1950).

Proof. Obviously $f^{-1}(\mathfrak{N}_\alpha)=\{f^{-1}(N)\mid N\in\mathfrak{N}_\alpha\}=\mathfrak{U}_\alpha$ is a normal[9] open covering of F for every $\alpha\in A$. Hence we can choose $\mathfrak{U}_{\alpha i}(i=1,2,\dots)$ from $\{\mathfrak{U}_\alpha\mid\alpha\in A\}$ such that $\mathfrak{U}_{\alpha 1}=\mathfrak{U}_\alpha$, $\mathfrak{U}_{\alpha 1}>\mathfrak{U}^*_{\alpha 2}>\mathfrak{U}_{\alpha 2}>\mathfrak{U}^*_{\alpha 3}>\cdots$. Putting $\mathfrak{V}'_{\alpha 1}=\{(R-F)\smile U\mid U\in\mathfrak{U}_{\alpha 3}\}$, we get a covering $\mathfrak{V}_{\alpha 1}=\mathfrak{V}'_{\alpha 1}\wedge\mathfrak{M}_{\alpha'}$ of R such that $\mathfrak{V}_{\alpha 1}\wedge F=\{V\frown F\mid F\in\mathfrak{V}_{\alpha 1}\}<\mathfrak{U}_{\alpha 2}$, $\mathfrak{V}_{\alpha 1}<\mathfrak{M}_{\alpha'}$. Since R is fully normal, we can choose further a covering $\mathfrak{V}_{\alpha 2}$ of R such that $\mathfrak{V}_{\alpha 2}\wedge F<\mathfrak{U}_{\alpha 3}$, $\mathfrak{V}^*_{\alpha 2}<\mathfrak{V}_{\alpha 1}$. We can obtain successively in the same way a sequence of coverings of R $\mathfrak{M}_{\alpha'}>\mathfrak{V}_{\alpha 1}>\mathfrak{V}^*_{\alpha 2}>\mathfrak{V}_{\alpha 2}>\mathfrak{V}^*_{\alpha 3}>\cdots$ such that $\mathfrak{V}_{\alpha i}\wedge F<\mathfrak{U}_{\alpha i+1}$ $(i=1,2,\dots)$.

Now we define a sequence of coverings of R from the above sequence by $\mathfrak{P}_{\alpha i}=\{\mathfrak{U}_{\alpha i},\ \mathfrak{V}_{\alpha i}\}=\{N(U,\mathfrak{V}_{\alpha i}),\ V\frown(R-F)\mid U\in\mathfrak{U}_{\alpha i},\ V\in\mathfrak{V}_{\alpha i}\}$, where $N(U,\mathfrak{V}_{\alpha i})$ denotes the open set $\smile\{V\mid\phi\neq V\frown F\subseteq U,\ V\in\mathfrak{V}_{\alpha i}\}$ of R. Let us show $\mathfrak{P}_{\alpha i}>\mathfrak{P}^{\Delta}_{\alpha i+1}(i=1,2,\dots)$. We denote by x an arbitrary point of R. If $S(x,\mathfrak{P}_{\alpha i+1})\frown F=\phi$, then there exists $V\in\mathfrak{V}_{\alpha i}$ such that $S(x,\mathfrak{P}_{\alpha i+1})=S(x,\mathfrak{V}_{\alpha i+1})\subseteq V$. Hence $S(x,\mathfrak{P}_{\alpha i+1})\subseteq V\frown(R-F)\in\mathfrak{P}_{\alpha i}$. If $x\in R-F$, $S(x,\mathfrak{P}_{\alpha i+1})\frown F\neq\phi$, then since $\mathfrak{V}^*_{\alpha i+1}<\mathfrak{V}_{\alpha i}$ and $\mathfrak{V}_{\alpha i}\wedge F<\mathfrak{U}_{\alpha i+1}$, there exist $V\in\mathfrak{V}_{\alpha i}$ and $U_0\in\mathfrak{U}_{\alpha i+1}$ such that $S(x,\mathfrak{V}_{\alpha i+1})\subseteq V$, $V\frown F\subseteq U_0\in\mathfrak{U}_{\alpha i+1}$. If $x\in N(U,\mathfrak{V}_{\alpha i+1})$, $U\in\mathfrak{U}_{\alpha i+1}$, then $U\frown U_0\neq\phi$, and hence from $\mathfrak{U}^*_{\alpha i+1}<\mathfrak{U}_{\alpha i}$ $S(U_0,\mathfrak{U}_{\alpha i+1})\subseteq U'$ for some $U'\in\mathfrak{U}_{\alpha i}$ and $V\frown F\subseteq U'$. Therefore $S(x,\mathfrak{V}_{\alpha i+1})\subseteq N(U',\mathfrak{V}_{\alpha i})$. Since $N(U,\mathfrak{V}_{\alpha i+1})\subseteq N(U',\mathfrak{V}_{\alpha i})$ is obvious, we obtain $S(x,\mathfrak{P}_{\alpha i+1})\subseteq N(U',\mathfrak{V}_{\alpha i})\in\mathfrak{P}_{\alpha i}$. If $x\in F$, then $S(x,\mathfrak{U}_{\alpha i+1})\subseteq U\in\mathfrak{U}_{\alpha i}$ for some U, and consequently $S(x,\mathfrak{P}_{\alpha i+1})\subseteq N(U,\mathfrak{V}_{\alpha i})\in\mathfrak{P}_{\alpha i}$. Therefore $\mathfrak{P}_{\alpha i}>\mathfrak{P}^{\Delta}_{\alpha i+1}$ is established.

Putting $(R-F)\smile S=T$, we define a mapping f^* from R into T by $f^*(z)=z(z\in R-F)$, $f^*(x)=f(x)\,(x\in F)$. Defining coverings $\mathfrak{Q}_{\alpha i}$ of T by $f^*(\mathfrak{P}_{\alpha i})=\mathfrak{Q}_{\alpha i}$, we have obviously $\mathfrak{Q}_{\alpha i}>\mathfrak{Q}^{\Delta}_{\alpha i+1}(i=1,2,\dots;\ \alpha\in A)$. Furthermore, $\{\mathfrak{Q}_{\alpha i}\wedge S\mid\alpha\in A;\ i=1,2,\dots\}=\{\mathfrak{N}_\alpha\mid\alpha\in A\}$ is obvious. If $x\in R-F$, then since F is closed, $S^2(x,\mathfrak{M}_{\alpha'})\frown F=\phi$ for some $\alpha'\in A'$, and consequently $S^2(x,\mathfrak{V}_{\alpha 1})\frown F=\phi$. Therefore $S(x,\mathfrak{P}_{\alpha 1})\frown F=\phi$, and $S(x,\mathfrak{Q}_{\alpha 1})\frown S=\phi$ is obvious. Hence S is a closed subset of T. Furthermore, if $x,\ y\in T$, $x\neq y$, then obviously $S(x,\mathfrak{Q}_{\alpha i})\not\ni y$ for some $\mathfrak{Q}_{\alpha i}$. Thus we can define a uniform topology in R by the uniform covering system $\{\wedge\{\mathfrak{Q}_{\alpha i}\mid(\alpha,i)\in C\}\mid C$ is a finite sub-set of $\{(\alpha,i)\mid\alpha\in A;\ i=1,2,\dots\}\}$ and obtain the uniform space T and the extension f^* of f satisfying conditions in this theorem.

The following Hausdorff's theorem is a special form of this theorem for $\mathfrak{m}=\mathfrak{a}$.

Hausdorff's theorem.[10] *If R and S are metric spaces, F is a*

9) A covering $\mathfrak{N}$ of R is called normal when there exists a sequence $\{\mathfrak{N}_i\mid i=1,2,\dots\}$ of coverings such that $\mathfrak{N}^*_{i+1}<\mathfrak{N}_i<\mathfrak{N}$ $(i=1,2,\dots)$.

10) F. Hausdorff: Erweiterung einer stetigen Abbildung, Fun. Math., **30** (1938). Recently, R. Arens gives a short proof of this theorem by a different method from us. R. Arens: Extension of functions on fully normal spaces, Pacific Journ. Math., **11** (1952).

closed set of R, and if f is a continuous mapping from F into S, then S can be imbedded isometrically in a metric space T such that f can be continuously extended to R with values in T, such that the extension is a homeomorphism of $R-F$ with $T-S$, and such that S is a closed sub-space of T. If f is a homeomorphism, then the extension is also a homeomorphism.

Lastly, let us discuss extension theorem in the case that R is not fully normal.

Theorem 5. *Theorem 4 is valid when R is normal and F satisfies the second countability axiom or when R is normal and S satisfies the second countability axiom.*

Proof. We assume that R is normal and F satisfies the second countability axiom and that $\{\mathfrak{N}_\alpha \mid \alpha \in A\}$ and $\{\mathfrak{M}_{\alpha'} \mid \alpha' \in A'\}$ are uniformities of S and R respectively. If we denote by f a continuous mapping on F having values in S, then $f^{-1}(\mathfrak{N}_\alpha)=\mathfrak{U}_\alpha$ is a normal covering of F. We choose coverings from $\{f^{-1}(\mathfrak{N}_\alpha) \mid \alpha \in A\}$ and take a sequence $\mathfrak{U}_\alpha=\mathfrak{U}_{\alpha 1}>\mathfrak{U}^*_{\alpha 2}>\mathfrak{U}_{\alpha 2}>\mathfrak{U}^*_{\alpha 3}>\cdots$ of coverings. Since F is regular and satisfies the second countability axiom, there exists a locally finite enumerable refinement $\mathfrak{U}=\{U_n \mid n=1, 2\ldots\}$ of $\mathfrak{U}_{\alpha 2}$. Let us denote by $\mathfrak{U}_0=\{U_{0n} \mid n=1, 2,\ldots\}$ a covering of F such that $\bar{U}_{0n}\subseteq U_n$, and consider continuous functions φ_n on R such that $\varphi_n(U_{0n})=1$, $\varphi_n(F-U_n)=0$, $0\leqq\varphi_n\leqq 1$. If we put $W_n=\{x \mid \varphi_n(x)>0\}$, then $\mathfrak{W}=\{W_n\}$ covers F, $\mathfrak{W}_\wedge F<\mathfrak{U}_{\alpha 2}$ and $\bigcup_{n=1}^{\infty} W_n=W\supseteq F$. Furthermore, we take a continuous function φ_0 on R such that $\varphi_0(W^C)=1$, $\varphi_0(F)=0$, $0\leqq\varphi_0\leqq 1$, and define $U_0=\{x \mid f_0(x)>0\}$. Then we have an enumerable covering $\mathfrak{W}_{\alpha 1}=\{U_0, U_1, U_2\ldots\}$ of R such that $\mathfrak{W}_{\alpha 1 \wedge}F<\mathfrak{U}_{\alpha 2}$. Since R is normal, from Corollary 2 $\mathfrak{W}_{\alpha 1}$ is a normal covering. Thus we have a normal covering $\mathfrak{V}_{\alpha 1}=\mathfrak{W}_{\alpha 1 \wedge}\mathfrak{M}_{\alpha'}$ of R such that $\mathfrak{V}_{\alpha 1 \wedge}F<\mathfrak{U}_{\alpha 2}$.

Next we take a normal covering $\mathfrak{T}_{\alpha 2}$ of R such that $\mathfrak{T}^*_{\alpha 2}<\mathfrak{T}_{\alpha 1}$, and a normal covering $\mathfrak{W}_{\alpha 2}$ of R such that $\mathfrak{W}_{\alpha 2 \wedge}F<\mathfrak{U}_{\alpha 3}$ in the same way as in the case of $\mathfrak{W}_{\alpha 1}$. Putting $\mathfrak{V}_{\alpha 2}=\mathfrak{W}_{\alpha 2 \wedge}\mathfrak{T}_{\alpha 1}$, we have a normal covering such that $\mathfrak{V}^*_{\alpha 2}<\mathfrak{V}_{\alpha 1}$, $\mathfrak{V}_{\alpha 2 \wedge}F<\mathfrak{U}_{\alpha 3}$. Repeating the above processes we obtain a sequence of uniform coverings $\mathfrak{V}_{\alpha 1}>\mathfrak{V}^*_{\alpha 2}>\mathfrak{V}_{\alpha 2}>\mathfrak{V}^*_{\alpha 3}>\cdots$ of R such that $\mathfrak{V}_{\alpha 1}<\mathfrak{M}_{\alpha'}$, $\mathfrak{V}_{\alpha i \wedge}F<\mathfrak{U}_{\alpha i+1}(i=1, 2\ldots)$ for every $\alpha \in A$. The remainder of the proof is the same as the proof of Theorem 4 and is omitted.