162. On the Commutativity of Projection Operators

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

Let P and Q be projection operators of closed linear manifolds M and N in a Hilbert space R, and $P\smile Q$ be a projection operator of the least closed linear manifold containing both M and N. It is known[1] that if P and Q are commutative: $PQ=QP$, then we have for every $x \in R$

$$\| P\smile Qx \|^2 \leqq \| Px \|^2 + \| Qx \|^2.$$

In this paper we shall show that this necessary condition for $PQ=QP$ is sufficient, too.

Theorem 1. For two projection operators P and Q, the following conditions are equivalent.

1) $PQ=QP$.

2) $\| P\smile Qx \|^p \leqq \| Px \|^p + \| Qx \|^p$ $(x \in R)$ *for all* p *in* $0<p\leqq 2$.

3) $\| P\smile Qx \|^p \leqq \| Px \|^p + \| Qx \|^p$ $(x \in R)$ *for some* p *in* $0<p\leqq 2$.

Proof. If $PQ=QP$, then we have $\| P\smile Qx \|^2 \leqq \| Px \|^2 + \| Qx \|^2$ $(x \in R)$. In the proof of 2) without loss of generality we can suppose $P\smile Q=I$ and $\| x \|=1$. Therefore, we have $\| P\smile Qx \|^p=1$ $=\| P\smile Qx \|^2 \leqq \| Px \|^2 + \| Qx \|^2 \leqq \| Px \|^p + \| Qx \|^p$ for all p in $0<p\leqq 2$. Next we prove that 3) implies 1). As $(P\smile Q)Px=Px$, $(P\smile Q)QPx$ $=QPx$, and $Px=QPx+(I-Q)Px$, we have $(P\smile Q)(I-Q)Px=(I-Q)Px$. Therefore, we conclude $\| (I-Q)Px \|^p = \| P\smile Q(I-Q)Px \|^p \leqq \| P(I-Q)Px \|^p + \| Q(I-Q)Px \|^p = \| P(I-Q)Px \|^p$, and hence $\| (I-Q)Px \| \leqq \| P(I-Q)Px \|$, so that $(I-Q)P=P(I-Q)P$, that is, $QP=PQP$. Therefore, we obtain $PQ=(QP)^*=(PQP)^*=PQP=QP$.

Theorem 2. For two projection operators P and Q, the following conditions are equivalent.

1) $P\leqq Q$ *or* $Q\leqq P$.

2) $\| P\smile Qx \| = \max \{\| Px \|, \| Qx \|\}$ $(x \in R)$.

3) $\| P\smile Qx \|^p \leqq \| Px \|^p + \| Qx \|^p$ $(x \in R)$ *for all* $p>2$.

4) $\| P\smile Qx \|^p \leqq \| Px \|^p + \| Qx \|^p$ $(x \in R)$ *for some* $p>2$.

Proof. It is evident that 1) implies 2), 2) implies 3), and 3) implies 4), because $\max \{\| Px \|, \| Qx \|\} \leqq (\| Px \|^p + \| Qx \|^p)^{1/p}$. We prove that 4) implies 1). By the similar method as in Theorem 1 we obtain $PQ=QP$ from 4). Therefore, if we have not 1) for P and Q, there are x_1, x_2, and x_3 such that $P(I-Q)x_1=x_1\neq 0$, $Q(1-P)x_2$

1) H. Nakano: Spectral theory in the Hilbert space, Tokyo Math. Book Series, **4** (1953), Theorem 12.7.

$=x_2\neq 0$, $x_1+x_2=x_3$, and $\|x_3\|=1$. Therefore, we have $P\smile Qx_3=x_3$, $Px_3=x_1$, $Qx_3=x_2$, and $\|x_1\|^2+\|x_2\|^2=1$, and hence $1=\|x_3\|^p=\|P\smile Qx_3\|^p \leqq \|Px_3\|^p+\|Qx_3\|^p=\|x_1\|^p+\|x_2\|^p<\|x_1\|^2+\|x_2\|^2=1$. Thus 4) implies 1).

Theorem 3. For two projection operators P and Q, $P=Q$ is equivalent to $\|P\smile Qx\|^2=\|Px\|\,\|Qx\|$ $(x\in R)$.

Proof. If $\|P\smile Qx\|^2=\|Px\|\,\|Qx\|$ $(x\in R)$, then we have $\|P\smile Qx\|^2 \leqq \frac{1}{2}(\|Px\|^2+\|Qx\|^2)$. As in the proof of Theorem 1, we obtain $\|(I-Q)Px\|^2\leqq\frac{1}{2}\|P(I-Q)Px\|^2\leqq\frac{1}{2}\|(I-Q)Px\|^2$ $(x\in R)$, therefore, $(I-Q)P=0$, that is, $P=QP$. Similarly we also obtain $Q=PQ$. Therefore we have $P=(QP)^*=PQ=Q$.