51. On Compact Semi-groups

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In this note, we shall extend a theorem of my paper [4], and apply the theorem to study the structure of compact semi-groups.[1]

A semi-group S of elements $a, b, c \cdots$ is called a *homogroup* if

1) S contains an idempotent e,
2) for each a of S, there exist elements a', a'' such that
$$aa' = e = a''a,$$
3) for every a of S,
$$ae = ea.$$

The terminology "homogroup" has been used by G. Thierrin [8], A. H. Clifford and D. D. Miller [1] have used the "semi-group having zeroid elements". In my paper [4], we proved the following

Theorem 1. Any compact commutative semi-group is homogroup.

The similar theorem has been also obtained by R. J. Koch [6].

Definition. A semi-group S is called *reversible* (following G. Thierrin [9]), if for any two elements a and b, $aS \frown bS \neq 0 \neq Sa \frown Sb$.

It is easily seen that *any commutative semi-group* or *any semi-group with zero-element is reversible.* We shall prove Theorem 2 which is a generalisation of Theorem 1.

Theorem 2. A compact semi-group is homogroup, if and only if it is reversible.

Such a theorem for finite semi-group has been proved by G. Thierrin [9].

Proof. Suppose that S is compact homogroup, then S contains an idempotent e, and, for any two elements a and b, there are two elements a', b' such that
$$aa' = e = bb' \qquad a', b' \in S.$$
Therefore $aS \frown bS \ni e$. Similarly $Sa \frown Sb \ni e$. This shows that $aS \frown bS$, $Sa \frown Sb$ are non-empty.

Conversely, suppose that S is reversible, if S contains zero-element 0, Theorem 2 is clear. Suppose that S does not contain zero-element 0. By the compactness of S, S contains at least one closed right minimal ideal A (for detail, see K. Iséki [4]). Suppose that B is a closed minimal right ideal of S different from A. Let

1) For general theory of semi-groups, see P. Dubreil [3].

a and b be elements in A and B respectively, then we have $aS \frown bS \neq 0$. Hence $AS \frown BS \neq 0$. This implies $A \frown B \neq 0$. By the minimality of A, B, $A = B$. S contains only one closed minimal right ideal A. By a theorem of A. H. Clifford [2], A is a minimal left ideal of S, since S does not contain zero-element. Let $a \in A$, then aA is a closed right ideal of S and $aA \subset A$. Hence $aA = A$.

Similarly, since A is two-sided ideal, $A \supset Aa$. By the minimality of A, $aA = A = Aa$ for a of A. Hence A is a subgroup of S. By a theorem of G. Thierrin [8], S is a homogroup.

From Theorem 2, we obtain Theorem 1 and the following

Corollary 1. A finite semi-group is homogroup, if and only if it is reversible.

Let S be a compact semi-group, and $\gamma(a)$ the set of positive powers of an element a of S. It is clear that $\overline{\gamma(a)}$ is a compact commutative semi-group. Therefore, by Theorem 1, $\overline{\gamma(a)}$ is a homogroup. Hence $\overline{\gamma(a)}$ contains only one idempotent and a compact commutative subgroup of S.

Theorem 3. Any compact semi-group contains at least one idempotent.[2]

Let e be an idempotent of S, and $K^{(e)}$ the set of elements $a \in S$ such that $e \in \overline{\gamma(a)}$. S. Schwarz [7] has proved that, for a compact commutative semi-group, each $K^{(e)}$ is a semi-group. We shall extend his result to strongly reversible semi-group, a new class: a semi-group S is said to be *strongly reversible*, if for any two elements a, b of S, there are natural numbers r, s and t such that

$$(ab)^r = a^s b^t = b^t a^s.$$

Similar theorem for strongly reversible periodic semi-group has proved in my Note [5].

Theorem 4. For strongly reversible compact group, each $K^{(e)}$ is a semi-group.

Proof. The idea of proof is due to S. Schwarz [7]. Let $\{U_\tau(e)\}$ be a complete system of neighbourhoods of e, then every $A_\tau = U_\tau(e) \frown \gamma(a^s)$ is non-empty. Let $A_\tau = \{a^{\tau_1 s}, a^{\tau_2 s}, \cdots\}$, and the set B_τ is defined by $\{b^{\tau_1 t}, b^{\tau_2 t}, \cdots\}$. We shall show $\bigcap_\tau \overline{B}_\tau \neq 0$. For any finite $B_{\tau(1)}, \cdots B_{\tau(k)}$, we take $A_{\tau(1)}, \cdots, A_{\tau(k)}$. For each $A_{\tau(i)}$, there is a neighbourhood $U_{\tau(i)}(e)$ of e. Then we can find a neighbourhood $U_\rho(e)$ such that

$$U_{\tau(1)} \frown \cdots \frown U_{\tau(k)} \supset U_\rho(e).$$

$e \in \overline{\gamma(a^s)}$ implies $A_\rho = U_\rho(e) \frown \{a^s, a^{2s}, \cdots\} = \{a^{\rho_1 s}, a^{\rho_2 s}, \cdots\}$. Let $B_\rho = \{b^{\rho_1 t}, b^{\rho_2 t}, \cdots\}$, then

2) Theorem 3 has been proved by some writers.

$$0 \neq B_\rho \subseteq B_{\tau(1)} \frown \cdots \frown B_{\tau(k)}.$$

Therefore $\{\overline{B}_\rho\}$ has the finite intersection property and this shows that $\bigcap_\tau \overline{B}_\tau$ is not empty, since S is compact. Take one element d of $\bigcap_\tau \overline{B}_\tau$. Since S is strongly reversible, we can find three positive integers u, v and w such that

$$(ed)^u = e^v d^w = d^w e^v$$
$$= ed^w = d^w e.$$

For any $U((ed)^u)$, we take $U_\sigma(e)$, $U(d^w)$ such that

$$U_\sigma(e) \cdot U(d^w) \subset U((ed)^u).$$

So we define

$$A_\sigma = U_\sigma(e) \frown \gamma(a^s) = \{a^{\sigma_1 s}, a^{\sigma_2 s}, \cdots\}$$
$$B_\sigma = \{b^{\sigma_1 t}, b^{\sigma_2 t}, \cdots\}.$$

Then $\overline{B}_\sigma \supset \bigcap_\tau \overline{B}_\tau \ni d$. Therefore we have $\overline{B}_\sigma \ni d^k$ and

$$0 \neq B_\sigma \frown U(d^w) = U(d^w) \frown \{b^{\sigma_1 t}, b^{\sigma_2 t}, \cdots\}.$$

From $b^{\sigma_i t} \in U(d^w) \frown B_\sigma$ and $a^{\sigma_i s} \in A_\sigma = \{a^{\sigma_1 s}, a^{\sigma_2 s}, \cdots\}$ and strongly reversibility of S, we have

$$(ab)^{r\sigma_i} = a^{s\sigma_i} b^{t\sigma_i} \in (U_\sigma(e) \frown A) \cdot (U(d^w) \frown B_\sigma)$$
$$\subset U_\sigma(e) \cdot U(d^w) \subset U((ed)^u).$$

Therefore we have $(ab)^{r\sigma_i} \in \overline{\gamma((ab)^r)}$. Hence $U((ed)^u) \frown \overline{\gamma((ab)^r)} \neq 0$ and we have $(ed)^u = ed^w \in \overline{\gamma((ab)^u)}$.

Next, we shall prove $e \in \overline{\gamma((ab)^r)}$. To prove this, let $D = \{ed^w, ed^{2w}, \cdots\}$, then $ed^w \in \overline{\gamma((ab)^r)}$ and ed^{kw} $(k = 1, 2, \cdots)$ are in $\overline{\gamma((ab)^r)}$. Hence $D \subset \overline{\gamma((ab)^r)}$ and $\overline{D} \subset \overline{\gamma((ab)^r)}$. For a given neighbourhood $U(e)$, we take $U'(e)$ such that $U'(e) U'(e) \subset U(e)$. From $\gamma(d^w) \subseteq \overline{\gamma(b^t)}$, we have $\overline{\gamma(d^w)} \subseteq \overline{\gamma(b^t)}$ and $e \in \overline{\gamma(d^w)}$. This shows that $U'(e) \frown \gamma(d^w) \neq 0$. Therefore, for some ρw,

$$d^{\rho w} \in U(e)$$

and

$$ed^{\rho w} \in U'(e) U'(e) \subset U(e).$$

Hence, we have $e \in \overline{D} \subset \overline{\gamma((ab)^r)}$. This completes the proof of Theorem 3.

From Theorem 3, we have the following

Theorem 4. If a compact semi-group S is strongly reversible, then each set $K^{(e)}$ is maximal semi-group, i.e. $K^{(e)}$ is the largest sub-semi-group of S containing only one idempotent e. S is the sum of disjoint semi-groups $K^{(e)}$.

References

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