162. On Interpolations of Analytic Functions. I
(Preliminaries)

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Walsh[1] has proved the following theorem: *Let* $f(z)$ *be a function single valued and analytic within the circle* $C_\rho : |z|=\rho>1$, *but not analytic regular on* C_ρ. *Then the sequence of polynomials* $P_n(z;f)$ *of respective degrees n found by interpolation to* $f(z)$ *in all the zeros of polynomials* $Z^{n+1}-1$ *converges to* $f(z)$ *throughout the interior of the circle* C_ρ, *uniformly on any closed set interior to* C_ρ *and diverges at every points exterior to* C_ρ *as n tends to infinity.* He has mentioned the possibility of a generalization of this theorem in his paper.

For the convergence of sequences of polynomials found by interpolations in sets of points which satisfy a certain condition, a complete result has been shown by Walsh,[2] but for the divergence, problems have been left unsolved.

For this divergence problem of such a sequence, a few works have been done by the author,[3)-5)] but these results were not satisfactory. But soon afterwards a little satisfactory result has been obtained by the author:[6]

Let the sequence of points

$$(\mathrm{P})\qquad \begin{cases} z_1^{(1)} \\ z_1^{(2)},\ z_2^{(2)} \\ z_1^{(3)},\ z_2^{(3)},\ z_3^{(3)} \\ \cdots\cdots\cdots \\ z_1^{(n)},\ z_2^{(n)},\ z_3^{(n)},\cdots,\ z_n^{(n)} \\ \cdots\cdots\cdots \end{cases}$$

which do not lie exterior to the unit circle $C:|z|=1$, *satisfy the condition that the sequence of*

1) J. L. Walsh: The divergence of sequences of polynomials interpolating in roots of unity, Bulletin Am. Math. Soc., **12**, 715 (1936).

2) J. L. Walsh: Interpolation and approximation, Am. Math. Soc. Coll. Publ., **20** (1935).

3) T. Kakehashi: On the convergence-region of interpolation polynomials, Jour. Math. Soc. Japan, **7**, 32 (1955).

4) T. Kakehashi: The divergence of interpolations. I-III, Proc. Japan Acad., **30**, Nos. 8,9,10 (1954).

5) T. Kakehashi: Integrations on the circle of convergence and the divergence of interpolations. I, Proc. Japan Acad., **31**, No. 6, 329 (1955).

6) T. Kakehashi: The decomposition of coefficients of power-series and the divergence of interpolation polynomials, Proc. Japan Acad., **31**, No. 8, 517 (1955).

$$\frac{W_n(z)}{z^n}=\frac{(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})}{z^n}$$

converges to a function $\lambda(z)$, single valued, analytic and non-vanishing for z exterior to C, and converges uniformly on any bounded closed points set exterior to C, that is

$$\lim_{n\to\infty}\frac{W_n(z)}{z^n}=\lambda(z) \qquad \textit{for } |z|>1.$$

Let the function $f(z)$ be single valued and analytic throughout the interior of the circle $C_\rho: |z|=\rho>1$ but not analytic regular on C_ρ. Then the sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $W_{n+1}(z)$ diverges at every point exterior to C_ρ. Moreover we have

$$\overline{\lim}_{n\to\infty}|P_n(z;f)|^{\frac{1}{n}}=\frac{|z|}{\rho} \qquad \textit{for } |z|>\rho.$$

In this paper, we shall consider a generalization of the result above-mentioned, and treat some applications.

1. In this paragraph, we consider some properties of coefficients obtained in the case when an analytic function is expanded into Laurent's series (or power-series). Let $f(z)$ be a function single valued and analytic on the region between two circles $C_\rho: |z|=\rho$ and $C_r: |z|=r<\rho$, but not analytic regular on C_ρ. Then the function $f(z)$ can be expanded into Laurent's series

$$(1) \qquad f(z)=\sum_{n=1}^{\infty}B_n\left(\frac{\rho}{z}\right)^n+\sum_{n=0}^{\infty}A_n\left(\frac{z}{\rho}\right)^n,$$

where coefficients A_n and B_n satisfy respectively

$$(2) \qquad \overline{\lim}_{n\to\infty}|A_n|^{\frac{1}{n}}=1 \quad\text{and}\quad \overline{\lim}_{n\to\infty}|B_n|^{\frac{1}{n}}\leqq\frac{r}{\rho}<1.$$

It has been proved by the author that the series (1) can be also represented[6] *by*

$$(3) \qquad f(z)=\sum_{n=-\infty}^{\infty}a_n\lambda_n\left(\frac{z}{\rho}\right)^n,$$

where

$$(4) \qquad \lambda_n=1 \qquad \textit{for } n=0,-1,-2,\cdots,$$

a_n satisfy

$$(5) \qquad \overline{\lim}_{n\to\infty}|a_n|=1,\quad \overline{\lim}_{n\to\infty}|a_{-n}|^{\frac{1}{n}}\leqq\frac{r}{\rho}<1,$$

and λ_n satisfy

$$(6) \qquad \lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}=1.$$

If we put $a_n=0: n=-1,-2,\cdots$ in the equation (3), we can consider the case when the function $f(z)$ is single valued and analytic throughout the interior of the circle C_ρ.

Next we consider several lemmas which show some properties of Laurent's series.

Lemma 1. *Let* $f(z)=\sum_{n=-\infty}^{\infty} a_n\lambda_n\left(\frac{z}{\rho}\right)^n$ *be the function which satisfies the conditions (4), (5) and (6), and* $\varphi(z)$ *be a function single valued, analytic and non-vanishing on* $C_\rho: |z|=\rho$. *If we put*

$$f(z)\varphi(z)=\sum_{n=-\infty}^{\infty} \gamma_n\left(\frac{z}{\rho}\right)^n, \tag{7}$$

the upper limit of $\frac{|\gamma_n|}{\lambda_n}$ *is bounded and positive, that is, we have*

$$\infty>\overline{\lim}_{n\to\infty}\frac{|\gamma_n|}{\lambda_n}>0. \tag{8}$$

If we put

$$\varphi(z)\equiv\sum_{n=-\infty}^{\infty}\alpha_n\left(\frac{z}{\rho}\right)^n,$$

we have

$$\gamma_n=\sum_{p=-\infty}^{\infty} a_p\lambda_p\alpha_{n-p}$$

and

$$\overline{\lim}_{p\to\infty}|\alpha_{\pm p}|^{\frac{1}{p}}\leqq R<1,$$

where R is a positive number less than unity determined by the situation of singularities of $\varphi(z)$. For any two integers n and p, and for any positive number δ, we can verify by the condition of λ_n and a_n that there exist two positive numbers A and B, independent of δ, n and p, which satisfy

$$\left|\frac{a_p\lambda_p}{\lambda_n}\right|\leqq A(1+\delta)^{|p-n|}$$

and

$$|\alpha_n|\leqq B(R+\delta)^{|n|}.$$

Accordingly, we have the following relations:

$$\begin{aligned}\frac{|\gamma_n|}{\lambda_n}&=\frac{|\sum_{p=-\infty}^{\infty}a_p\lambda_p\alpha_{n-p}|}{\lambda_n}\\ &\leqq\left|\sum_{p=0}^{\infty}\frac{a_p\lambda_p}{\lambda_n}\alpha_{n-p}\right|+\left|\sum_{p=1}^{\infty}\frac{a_{-p}\lambda_{-p}}{\lambda_n}\alpha_{n+p}\right|\\ &\leqq AB\sum_{p=0}^{\infty}(1+\delta)^{|n-p|}(R+\delta)^{|n-p|}+AB\sum_{p=1}^{\infty}(1+\delta)^{|n+p|}(R+\delta)^{|n+p|}\\ &\leqq 2AB\sum_{p=0}^{\infty}(1+\delta)^p(R+\delta)^p.\end{aligned}$$

The last side is convergent for δ sufficiently small by the condition $R<1$. Hence we can verify that $\frac{|\gamma_n|}{\lambda_n}$ are uniformly bounded for any integer n. Then the relation $\overline{\lim}_{n\to\infty}\frac{|\gamma_n|}{\lambda_n}<\infty$ follows at once.

Next we shall prove the relation $\overline{\lim}_{n\to\infty}\frac{|\gamma_n|}{\lambda_n}>0$. If we put

$$\frac{1}{\varphi(z)}\equiv\sum_{n=-\infty}^{\infty}\beta_n\left(\frac{z}{\rho}\right)^n$$

which is single valued and analytic on C_ρ, we have

$$\overline{\lim}_{n\to\infty}|\beta_n|^{\frac{1}{n}}<1 \quad\text{and}\quad \overline{\lim}_{n\to\infty}|\beta_{-n}|^{\frac{1}{n}}<1.$$

From the equation

$$f(z)=\frac{f(z)\varphi(z)}{\varphi(z)}=\sum_{n=-\infty}^{\infty}a_n\lambda_n\left(\frac{z}{\rho}\right)^n=\sum_{n=-\infty}^{\infty}\left(\sum_{p=-\infty}^{\infty}\gamma_{n-p}\beta_p\right)\left(\frac{z}{\rho}\right)^n,$$

we have

$$a_n=\frac{1}{\lambda_n}\sum_{p=-\infty}^{\infty}\gamma_{n-p}\beta_p=\sum_{p=-\infty}^{\infty}\frac{\lambda_{n-p}}{\lambda_n}\frac{\gamma_{n-p}}{\lambda_{n-p}}\beta_p,$$

where $\lambda_k=1$ for $k\leqq 0$.

If we assume $\lim_{n\to\infty}\frac{\gamma_n}{\lambda_n}=0$, $\max_{-\infty<n<\infty}\frac{|\gamma_n|}{\lambda_n}\equiv M$ exists for any integer (positive or negative) n. For any two integers n and p, and for any positive number δ, we can verify that there exists a positive number K, independent of n, p and δ, which satisfies

$$(9)\qquad \frac{\lambda_{n-p}}{\lambda_n}\leqq K(1+\delta)^{|p|}.$$

Accordingly, we have

$$\begin{aligned}|a_n|&=\left|\sum_{p=-\infty}^{\infty}\frac{\lambda_{n-p}}{\lambda_n}\frac{\gamma_{n-p}}{\lambda_{n-p}}\beta_p\right|\\&\leqq\max_{p\leqq 0}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=-\infty}^{0}\frac{\lambda_{n-p}}{\lambda_n}|\beta_p|+\max_{1\leqq p\leqq m}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=1}^{m}\frac{\lambda_{n-p}}{\lambda_n}|\beta_p|\\&\quad+M\sum_{p=m+1}^{\infty}\frac{\lambda_{n-p}}{\lambda_n}|\beta_p|\\&\leqq K\max_{q\geqq 0}\frac{|\gamma_{n+q}|}{\lambda_{n+q}}\sum_{q=0}^{\infty}|\beta_{-q}|(1+\delta)^q+K\max_{1\leqq p\leqq m}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=1}^{m}(1+\delta)^p|\beta_p|\\&\quad+KM\sum_{p=m+1}^{\infty}|\beta_p|(1+\delta)^p,\end{aligned}$$

where we can choose a positive number δ such that $\overline{\lim}_{n\to\infty}|\beta_{\pm n}|^{\frac{1}{n}}<\frac{1}{1+\delta}$. For any positive number ε, if we take m sufficiently large, the last term becomes less than $\frac{\varepsilon}{3}$. And for a fixed number m, if we take n sufficiently large, the first and the second terms become respectively less than $\frac{\varepsilon}{3}$ by the assumption $\lim_{n\to\infty}\frac{\gamma_n}{\lambda_n}=0$. Hence we have $\lim_{n\to\infty}a_n=0$ which contradicts the assumption $\overline{\lim}_{n\to\infty}|a_n|=1$. Thus the lemma is established.

Lemma 2. *Let* $f(z)=\sum_{n=-\infty}^{\infty}a_n\lambda_n\left(\frac{z}{\rho}\right)^n$ *be the function which satisfies* (4), (5) *and* (6), *and* $\varphi_n(z)$; $n=1,2,\cdots$ *be a sequence of functions, single valued and analytic on a closed domain* G *which contains the* *circle C_ρ in its interior, such that*

(10) $\lim_{n\to\infty}\varphi_n(z)=0$ *uniformly on the domain G.*

If we put

(11) $$f(z)\varphi_n(z)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}\left(\frac{z}{\rho}\right)^k.$$

Then we have

(12) $$\lim_{n\to\infty}\frac{\gamma_n^{(n)}}{\lambda_n}=0.$$

If we put

$$\varphi_n(z)=\sum_{k=-\infty}^{\infty}\alpha_k^{(n)}\left(\frac{z}{\rho}\right)^k,$$

we can choose a positive number δ_0 such that $\varphi_n(z)$ are single valued and analytic on and between two circles $C_{(1+\delta_0)\rho}:|z|=(1+\delta_0)\rho$ and $C_{(1+\delta_0)^{-1}\rho}:|z|=\dfrac{\rho}{1+\delta_0}$, and we have

$$\alpha_k^{(n)}=\frac{\rho^k}{2\pi i}\int_{C_{(1+\delta_0)\rho}}\varphi_n(t)t^{-k-1}dt\ ;\quad k=0,1,2,\cdots,$$

$$\alpha_k^{(n)}=\frac{\rho^k}{2\pi i}\int_{C_{(1+\delta_0)^{-1}\rho}}\varphi_n(t)t^{-k-1}dt\ ;\quad k=-1,-2,\cdots.$$

Accordingly, we can verify that, for any integer k, the relation

(13) $$|\alpha_k^{(n)}|\leqq M_n(1+\delta_0)^{-k}$$

holds for any integer n, where M_n can be allowed to approach zero as n tends to infinity.

From the equation

$$\gamma_n^{(n)}=\sum_{p=-\infty}^{\infty}a_p\lambda_p\alpha_{n-p}^{(n)},$$

we have

$$\frac{\gamma_n^{(n)}}{\lambda_n}=\sum_{p=-\infty}^{\infty}a_p\frac{\lambda_p}{\lambda_n}\alpha_{n-p}^{(n)}=\sum_{q=-\infty}^{\infty}a_{n-q}\frac{\lambda_{n-q}}{\lambda_n}\alpha_q^{(n)}$$

$$=\sum_{q=0}^{\infty}a_{n+q}\frac{\lambda_{n+q}}{\lambda_n}\alpha_{-q}^{(n)}+\sum_{q=1}^{\infty}a_{n-q}\frac{\lambda_{n-q}}{\lambda_n}\alpha_q^{(n)}.$$

For any positive number δ less than δ_0, if we put $M\equiv\max|a_n|$, we have

$$\frac{|\gamma_n^{(n)}|}{\lambda_n}\leqq MKM_n\sum_{q=0}^{\infty}\left(\frac{1+\delta}{1+\delta_0}\right)^q+MKM_n\sum_{q=1}^{\infty}\left(\frac{1+\delta}{1+\delta_0}\right)^q$$

by (9) and (13), where M and K are respectively independent of δ, δ_0, n and q. And as M_n can be allowed to approach zero, $\dfrac{\gamma_n^{(n)}}{\lambda_n}$ clearly tends to zero as n tends to infinity. Thus the lemma is established.

The following lemma follows at once from Lemmas 1 and 2.

Lemma 3. *Let $f(z)=\sum_{n=-\infty}^{\infty}a_n\lambda_n\left(\frac{z}{\rho}\right)^n$ be the function which satisfies (4), (5) and (6), and $\varphi_n(z):n=1,2,\cdots$ be the sequence of functions,* *single valued and analytic on C_ρ, such that*

(14) $$\lim_{n\to\infty}\varphi_n(z)=\varphi(z) \qquad (\textit{non-vanishing on } C_\rho)$$

uniformly on a closed domain which contains the circle C_ρ in its interior. If we put

(15) $$f(z)\varphi(z)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}\left(\frac{z}{\rho}\right)^k,$$

then we have

(16) $$\infty>\overline{\lim}_{n\to\infty}\frac{|\gamma_n^{(n)}|}{\lambda_n}>0.$$