36. *A Remark on the Ranked Space. II*[1]

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1. We have proved the Baire's theorem with respect to the changed topology[2] in the case that the system of neighbourhoods of a space does not satisfy the axiom (C) of F. Hausdorff, while there is an example[3] of a ranked space which is complete with respect to the given topology but not with respect to the changed topology. Therefore, it is desirable to show the Baire's theorem with respect to the given topology. The purpose of this paper is to do this by modifing the definition of non-density in the case that the system of neighbourhoods does not satisfy the axiom (C).

Definitions. Let R be a space having a system of neighbourhoods which satisfies the axiom (A) of F. Hausdorff, and denote by $\mathfrak{T}$ its topology. Excepting the two extreme cases,[4] we shall modify the definition of the depth $\omega(R, p)$ of a point p. The monotone decreasing sequence of neighbourhoods

(1) $\quad v_0(p_0) \supseteq v_1(p_1) \supseteq \cdots \supseteq v_\alpha(p_\alpha) \supseteq \cdots; \; 0 \leq \alpha < \beta,$

will be called of *type* β, and the sequence (1) will be called *maximal* if there exists no neighbourhood $v(p)$ such that $v(p) \subseteq \bigcap_{\alpha<\beta} v_\alpha(p_\alpha)$. The sequence of neighbourhoods of a fixed point p

(2) $\quad v_0(p) \supseteq v_1(p) \supseteq \cdots \supseteq v_\alpha(p) \supseteq \cdots; \; \alpha < \beta,$

will be called *maximal at the point* p if there exists no neighbourhood $v(p)$ of the point p such that $v(p) \subseteq \bigcap_{\alpha<\beta} v_\alpha(p)$. It is clear that the sequence (2), which is maximal at the point p, is not necessarily to be maximal in the general meaning. Let $v(p)$ be an arbitrary neighbourhood of p, and denote by $\omega(R, v(p))$ the minimum ordinal number of the types of maximal sequences having $v(p)$ as its first term, and we shall call $\omega(R, v(p))$ the *depth of* $v(p)$. Put into $\omega(R, p) = \inf_{v(p)} \omega(R, v(p))$ and we shall call $\omega(R, p)$ *the depth of the point* p. The depth $\omega(R)$[5] of the space R and the ranked space[6] shall be defined by using the new $\omega(R, p)$.

It must be remarked that the depth defined by Prof. K. Kunugi

1) We shall hereafter translate "l'espace rangé" into "the *ranked* space".
2) T. Shirai: A remark on the ranged space, Proc. Japan Acad., **32** (1956).
3) See Example at the end of this paper.
4) K. Kunugi: Sur les espaces complets et régulièrement complets. I, Proc. Japan Acad., **30** (1954).
5) K. Kunugi: *Op. cit.*
6) K. Kunugi: *Op. cit.*

is uniquely determined by the topology of the space and that the new depth is a notion determined by the system of neighbourhoods of the space but not by the topology of it. We also remark that the new depth $\omega(R)$ is an inaccessible limit ordinal number which is, in general, greater than the depth defined by Prof. K. Kunugi.

Denote by $\overline{M}$, the closure of a subset $M \subseteq R$ with respect to the given topology $\mathfrak{T}$, and denote by $\widetilde{M}$ the set of all points p such that $I(v(p)) \cap M \neq \phi$ for every neighbourhood $v(p)$ of p, where $I(v(p))$ means the interior of $v(p)$ with respect to the given topology $\mathfrak{T}$.

Lemma 1. *Suppose that the system of neighbourhoods satisfies the axiom (B) of F. Hausdorff. If $\widetilde{I(A)}=R$, then we have $I(A\cap V(x))\neq\phi$, for an arbitrary point x ($\in R$) and an arbitrary neighbourhood $V(x)$ of x.*

Proof. Since $\mathfrak{T}$ is distributive by hypothesis, it follows that

$$I(A\cap V)=((\overline{A\cap V})^c)^{c\,7)}=(\overline{A^c\cup V^c})^c=(\overline{A^c}\cup\overline{V^c})^c=(\overline{A^c})^c\cap(\overline{V^c})^c$$
$$=I(A)\cap I(V).$$

Therefore, we have this lemma.

Lemma 2. *For every sequence of type $\beta<\omega(R)$*

(1) $\qquad v_0(p_0)\supseteq v_1(p_1)\supseteq\cdots\supseteq v_\alpha(p_\alpha)\supseteq\cdots;\ \alpha<\beta,$

we have

$$I(\bigcap_{\alpha<\beta} v_\alpha(p_\alpha))\neq\phi.$$

Proof. Since $\beta<\omega(R)\leq\omega(R, v_0(p_0))$, the sequence (1) must not be maximal, and so we have this lemma immediately.

Lemma 3. *Suppose that the system of neighbourhoods satisfies the axiom (B) of F. Hausdorff. For every sequence of sets*

(1) $\qquad A_0, A_1, A_2,\cdots, A_\alpha,\cdots;\ \alpha<\omega_\nu\leq\omega(R),$

where

(2) $\qquad \widetilde{I(A_\alpha)}=R,\quad \alpha<\omega_\nu$

and ω_ν is an inaccessible limit number, there exists a fundamental sequence

(3) $\qquad v_0(p_0)\supseteq v_1(p_1)\supseteq\cdots\supseteq v_\alpha(p_\alpha)\supseteq\cdots;\ \alpha<\omega_\nu,$

where $v_\alpha(p_\alpha)\in\mathfrak{B}_{\gamma_\alpha}$, $\gamma_0<\gamma_1<\cdots<\gamma_\alpha<\cdots$ and $p_{2\alpha+1}=p_{2\alpha}$, having the following property

$$v_{2\alpha}(p_{2\alpha})\subseteq(A_{2\alpha}\cap V(x))$$

where x is an arbitrary given fixed point and $V(x)$ is an arbitrary given fixed neighbourhood of x.

Proof. Since $\widetilde{I(A_0)}=R$, it follows that $I(A_0\cap V)\neq\phi$, *i.e.* there exist a point p and a neighbourhood $v_0(p_0)$ such that $v_0(p_0)\subseteq(A_0\cap V)$. Put into γ_0 the rank of $v_0(p_0)$.

Suppose that we have defined already the sequence of neighbourhoods

7) M^c means the complement of a set M with respect to the space R.

$$v_0(p_0) \supseteq v_1(p_1) \supseteq \cdots \supseteq v_\alpha(p_\alpha) \supseteq \cdots; \ \alpha < \beta \ (< \omega_\nu)$$

having above properties.

In the case β is an even number (including limit number). Put into

$$B^{(\beta)} = \bigcap_{\alpha < \beta} v_\alpha(p_\alpha),$$

since $\beta < \omega_\nu \leq \omega(R)$, by virtue of Lemma 2, it follows that there exist a point $q \in I(B^{(\beta)})$ and a neighbourhood $V(q)$ such that $V(q) \subseteq B^{(\beta)}$. By virtue of Lemma 1, we have $I(A_\beta \cap V(q)) \neq \phi$, *i.e.* there exist a point p_β and a neighbourhood $v(p_\beta)$ such that $v(p_\beta) \subseteq (A_\beta \cap V(q))$. Since R is a ranked space, there exist a rank γ_β and a neighbourhood $v_\beta(p_\beta)$ of rank γ_β such that $v_\beta(p_\beta) \subseteq v(p_\beta)$, where $\gamma_\alpha < \gamma_\beta < \omega_\nu$ for every α $(< \beta)$. Then we have

$$v_\beta(p_\beta) \subseteq v(p_\beta) \subseteq (A_\beta \cap V(q)) \subseteq (A_\beta \cap B^{(\beta)}) = A_\beta \cap v_0(p_0) \subseteq A_\beta \cap V(x).$$

In the case β is odd. Put into $p_\beta = p_{\beta-1}$. Since $p_\beta = p_{\beta-1} \in I(v_{\beta-1}(p_{\beta-1}))$, there exists a neighbourhood $v(p_\beta)$ such that $v(p_\beta) \subseteq v_{\beta-1}(p_{\beta-1})$. Since R is a ranked space, there exist a rank γ_β $(> \gamma_{\beta-1})$ and a neighbourhood $v_\beta(p_\beta)$ of rank γ_β such that $v_\beta(p_\beta) \subseteq v(p_\beta)$ $(\subseteq v_{\beta-1}(p_{\beta-1}))$.

Definition. We shall call as usual that M is dense in R if $\overline{M} \supseteq R$. And we shall call that M is *non-dense* in R if $\widetilde{(M)^c} \supseteq R$. It is clear that when the system of neighbourhoods satisfies the axiom (C) of F. Hausdorff, this definition of non-density coincides with that of usual meaning. By using this new definition of non-density, we have the Baire's theorem with respect to the given topology.

Baire's theorem. *Suppose that the system of neighbourhoods satisfies the axiom (B) of F. Hausdorff. In the complete ranked space*[8] *R, every non empty set G must be of the 2nd category.*[9]

Proof. Suppose a non empty open set G can be of the 1st category:

$$G = \bigcup_{\alpha < \omega_\nu} B_\alpha, \quad \text{where } \widetilde{(\overline{B}_\alpha)^c} \supseteq R.$$

Put into $A_{2\alpha+1} = A_{2\alpha} = (B_\alpha)^c$, since B_α is non-dense, it follows that $\widetilde{I(A_\alpha)} = R$. For an arbitrary point x of R and an arbitrary neighbourhood $V(x)$, there exists, by virtue of Lemma 3, a fundamental sequence

$$(1) \qquad v_0(p_0) \supseteq v_1(p_1) \supseteq \cdots \supseteq v_\alpha(p_\alpha) \supseteq \cdots; \ \alpha < \omega_\nu,$$

where $v_\alpha(p_\alpha) \in \mathfrak{B}_{\gamma_\alpha}$, $p_{2\alpha+1} = p_{2\alpha}$ and $\gamma_0 < \gamma_1 < \cdots < \gamma_\alpha < \cdots$, having the following property

$$v_{2\alpha}(p_{2\alpha}) \subseteq (A_{2\alpha} \cap V(x)).$$

Since the space R is complete by hypothesis, it follows that

$$\phi \neq \bigcap_{\alpha < \omega_\nu} v_\alpha(p_\alpha) \subseteq \bigcap_{\alpha < \omega_\nu} v_{2\alpha}(p_{2\alpha}) \subseteq (\bigcap_{\alpha < \omega_\nu} A_{2\alpha}) \cap V(x).$$

8) K. Kunugi: *Op. cit.*

9) K. Kunugi: *Op. cit.*

This means that $\bigcap_{\alpha<\omega_\nu} A_{2\alpha}$ is dense in R. Since G is open, it follows that

$$G^c=\overline{G^c}=(\overline{\bigcup_{\alpha<\omega_\nu} B_\alpha})^c=(\overline{\bigcap_{\alpha<\omega_\nu} B_\alpha^c})=\overline{\bigcap_{\alpha<\omega_\nu} A_{2\alpha}}=R.$$

This contradicts the hypothesis that G is non empty.

Example. Let l be a sufficiently large fixed positive number, and let x, y, be the rectangular coordinates of the Euclidean plane.

Let $V(\lambda, n; 0)$, $n=1,2,3,\cdots$, $\lambda\geq l$, be the sets of all points $P(x,y)$ such that $xy<1/n$, $0\leq x<\lambda$ and $0\leq y<\lambda$. Denote by $V(\lambda, n; P)$ the set of all points $Q(\xi,\eta)$ such that $(\xi-x, \eta-y)\in V(\lambda, n; 0)$ where $P=(x,y)$. And put into $\mathfrak{B}_{n-1}$ the system of $V(\lambda, n; P)$ where $\lambda\geq l$ and P wipes out the Euclidean plane. Then the family of the systems $\mathfrak{B}_{n-1}$ ($n=1,2,3,\cdots$) defines a topology $\mathfrak{T}$ and a ranked space R of the depth ω. Of course, the system of neighbourhoods does not satisfy the axiom (C) of F. Hausdorff. Consider an arbitrary fundamental sequence

$$(1)\qquad v_0(P_0)\supseteq v_1(P_1)\supseteq\cdots\supseteq v_n(P_n)\supseteq\cdots$$

where $v_n(P_n)\in\mathfrak{B}_{\gamma_n}$, $v_n(P_n)=V(l+(1/\gamma_n+1), \gamma_n+1; P_n)$. Since

$$(2)\qquad P_n\in I(v_n(P_n))\subseteq I(v_m(P_m))\qquad\text{for } m\leq n,$$

the set of the points $P_0, P_1,\cdots, P_n\cdots$ has an accumulated point Q with respect to the Euclidean topology, and Q will be contained in the interior or the frontier (with respect to the Euclidean topology) of $I(v_0(P_0))$. Suppose that $Q\notin v_m(P_m)$, then the Euclidean distance between Q and the set $I(v_m(P_m))$ is positive and so there exists a number n ($>m$) such that $P_n\notin I(v_m(P_m))$. This contradicts (2). Therefore, *the ranked space R is complete.* On the other hand, this space *R is not strongly complete.*[10] In fact, there exists a fundamental sequence

$$v_0(P_0)\supseteq v_1(P_1')\supseteq v_2(P_2')\supseteq\cdots\supseteq v_n(P_n')\supseteq\cdots,$$

where $P_{2n+1}'=P_{2n}'=((1-1/2n)x_0+(1/2n)\xi, y_0)$ and (ξ, y_0) is the right lower angular point of the frontier (with respect to the Euclidean topology) of $I(v_0(P_0))$. Then the sequence

$$I(v_0(P_0))\supseteq I(v_1(P_1'))\supseteq\cdots\supseteq I(v_nP_n'))\supseteq\cdots$$

is a fundamental sequence with respect to the changed topology $\mathfrak{T}^*$ and $\bigcap_n I(v_n(P_n'))=\phi$.

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10) See 2).