35. On a Right Inverse Mapping of a Simplicial Mapping

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1. Let X and Y be topological spaces and let f be a continuous mapping from X onto Y. By a right inverse mapping of f, we mean a continuous mapping g of Y into X such that $fg(y)=y$ for each point y of Y. In the present note, we shall show that, in case X and Y are (finite or infinite) simplicial complexes and f is a simplicial mapping from X onto Y, the existence of a right inverse mapping of f is equivalent to some combinatorial properties of X and Y. The theorem will be stated in 3. In 2 we shall state notations and a lemma which we need later on.

2. We denote by J the additive group of integers. By a *lower sequence* of abelian groups, we mean sequences of abelian groups $\{G_i; i \in J\}$ and homomorphisms $\{g_i; i \in J\}$ such that

i) g_i is a homomorphism of G_{i+1} into G_i, $i \in J$;

ii) $g_i g_{i+1}$ is the zero-homomorphism, $i \in J$.

By a *homomorphism* of a lower sequence $\{G_i; g_i\}$ of abelian groups into a lower sequence $\{H_i; h_i\}$ of abelian groups, we mean a sequence $\{f_i; i \in J\}$ of homomorphisms such that

i) f_i is a homomorphism of G_i into H_i, $i \in J$;

ii) $h_i f_{i+1} = f_i g_i$, $i \in J$.

A homomorphism $\{f_i\}$ of a lower sequence $\{G_i; g_i\}$ into a lower sequence $\{H_i; h_i\}$ is called a *retraction-homomorphism* if and only if there exists a homomorphism $\{k_i\}$ of $\{H_i; h_i\}$ into $\{G_i; g_i\}$ such that, for each integer $i \in J$, $f_i k_i$ is the identity isomorphism of H_i into H_i.

Let X be a simplicial complex. We denote the i-section of X by X^i. Let A be a subcomplex of X. By the barycentric subdivision of X relative to A, we mean the barycentric subdivision of X such that all simplexes of A are not subdivided (cf. [1] or [3]).

Lemma. *Let X and Y be simplicial complexes and let f be a simplicial mapping of X into Y. Let B be a subcomplex of Y. Let us denote the first barycentric subdivisions of X and Y relative to the subcomplexes $f^{-1}(B)$ and B by $\widetilde{X}$ and $\widetilde{Y}$, respectively. Then there exists a simplicial mapping $\widetilde{f}$ of $\widetilde{X}$ into $\widetilde{Y}$, which we call a simplicial mapping associated with f and B with the following property: Let s and s' be simplexes of $X-f^{-1}(B)$ and $Y-B$. Then we have $f(s)=s'$ if and only if the barycenter of s is mapped into the barycenter of s' by $\widetilde{f}$.*

This lemma is obvious by the definition of a simplicial mapping.

Let (X, A) be a pair of simplicial complexes. We denote by $H_i(X, A)$ and $H_i(X)$ the i-dimensional homology groups of (X, A) and X with coefficients J. The sequence of groups and homomorphisms

$$\cdots \xleftarrow{i_*} H_{q-1}(A) \xleftarrow{\partial} H_q(X, A) \xleftarrow{j_*} H_q(X) \xleftarrow{i_*} H_q(A) \xleftarrow{\partial} \cdots$$

is a lower sequence, where i_* and j_* are the homomorphisms induced by the inclusion mappings $i: A \to X$ and $j: X \to (X, A)$, and ∂ is the boundary homomorphism (cf. for example, [2]). This sequence is called the homology sequence of (X, A). We denote it by $\mathcal{H}(X, A)$.

3. **Theorem.** *Let X and Y be (finite or infinite) simplicial complexes. Let f be a simplicial mapping from X onto Y. The following three conditions are equivalent:*

i) *There exists a simplicial mapping of Y into X which is a right inverse mapping of f.*

ii) *Let $\tilde{X}$ be the first barycentric subdivision of X relative to the subcomplex $f^{-1}(Y^0)$ of X and let $\tilde{Y}$ be the first barycentric subdivision of Y. Moreover, let $\tilde{f}$ be a simplicial mapping of $\tilde{X}$ into $\tilde{Y}$ associated with f and Y^0. Whenever (K, L) is a pair of subcomplexes of $\tilde{Y}$, $\tilde{f}$ induces a retraction-homomorphism from $\mathcal{H}(\tilde{f}^{-1}(K) \frown \tilde{X}^s, \tilde{f}^{-1}(L) \frown \tilde{X}^t)$ onto $\mathcal{H}(K, L)$, where $s = \dim K$ and $t = \dim L$.*

iii) *Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{f}$ be the same as in* ii). *Then $\tilde{f}$ induces a retraction-homomorphism from $\mathcal{H}(\tilde{X}^1, \tilde{X}^0)$ onto $\mathcal{H}(\tilde{Y}^1, \tilde{Y}^0)$.*

Proof. Since ii)$\to$iii) is obvious, it is sufficient to prove that i)$\to$ii) and iii)$\to$i).

i)$\to$ii). Let g be a right inverse simplicial mapping of f from Y to X. Let (K, L) be a pair of subcomplexes of $\tilde{Y}$ such that $s = \dim K$ and $t = \dim L$. Put $M = g(K)$ and $N = g(L)$. Obviously, $\tilde{f} \mid M = f \mid M$ and the restricted mapping $\tilde{f} \mid M : M \to K$ is a homeomorphism. Denote by h the restricted mapping $g\tilde{f} \mid \tilde{f}^{-1}(K) \frown \tilde{X}^s : (\tilde{f}^{-1}(K) \frown \tilde{X}^s, \tilde{f}^{-1}(L) \frown \tilde{X}^t) \to (M, N)$. Then h is a simplicial retraction from $(\tilde{f}^{-1}(K) \frown \tilde{X}^s, \tilde{f}^{-1}(L) \frown \tilde{X}^t)$ onto (M, N).*) Therefore, h induces a retraction-homomorphism from $\mathcal{H}(\tilde{f}^{-1}(K) \frown \tilde{X}^s, \tilde{f}^{-1}(L) \frown \tilde{X}^t)$ onto $\mathcal{H}(M, N)$. This completes the proof.

iii)$\to$i). Put $M = \tilde{X}^1$, $N = \tilde{X}^0$, $K = \tilde{Y}^1$ and $L = \tilde{Y}^0$. By our assumptions, there exist homomorphism $k_0 : H_0(L) \to H_0(N)$ and $k_1 : H_1(K, L) \to H_1(M, N)$ such that

a) $f_i k_i =$ the identity isomorphism for $i = 0, 1$;

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*) Let (A, B) and (C, D) be pairs of simplicial complexes such that $(C, D) \subset (A, B)$. By a simplicial retraction h from (A, B) onto (C, D), we mean a simplicial mapping from (A, B) onto (C, D) such that $h(x) = x$ for each point x of C.

b) $\partial k_1 = k_0 \partial$;

where f_0 is the homomorphism of $H_0(N)$ into $H_0(L)$ induced by $\tilde{f}$ and f_1 is the homomorphism of $H_1(M, N)$ into $H_1(K, L)$ induced by $\tilde{f}$. Since N and L are sets of vertexes, we have $H_0(N) = \sum_{v \in N} J_v$ and $H_0(L) = \sum_{w \in L} J_w$, where $\sum$ means the weak direct sum of abelian groups J_v and J_w each of which is isomorphic to J, respectively. Denote by 1_v and 1_w the unit elements of J_v and J_w for each vertex v and w of N and L. For each vertex w of L, we can find the vertex v of N such that $k_0(1_w) = 1_v$. Let $\tilde{g}_0$ be a mapping of L into N defined by $\tilde{g}_0(w) = v$. Then $\tilde{f}\tilde{g}_0(w) = w$ and $\tilde{g}_0$ is a 1-1 correspondence. Let $s = (w_0, w_1)$ be a 1-simplex of K. By b), $\tilde{g}_0(w_0)$ and $\tilde{g}_0(w_1)$ form a 1-simplex of M. Let $s = (w_0, w_1, \cdots, w_n)$ be an n-simplex of $\tilde{Y}$ such that w_i is the barycenter of a j_i-simplex of Y for $i = 0, 1, 2, \cdots, n$ and $j_i < j_{i+1}$ for $i = 0, 1, \cdots, n-1$. Let $\tilde{g}_0(w_i)$ be the barycenter of a j_i'-simplex $t_{j_i'}$ of X. Then, by the lemma in 2, we have $j_i \leq j_i'$, $i = 0, 1, \cdots, n$, and $j_i' < j_{i+1}'$, $i = 0, 1, \cdots, n-1$. Moreover $t_{j_i'}$ is a j_i'-face of $t_{j_{i+1}'}$, $i = 0, 1, \cdots, n-1$. Therefore the set $\{\tilde{g}_0(w_i);\ i = 0, 1, \cdots, n\}$ forms an n-face of $t_{j_n'}$. Thus we have a simplicial mapping $\tilde{g}$ of $\tilde{Y}$ into $\tilde{X}$ defined by $\tilde{g} \mid L = \tilde{g}_0$ such that $\tilde{f}\tilde{g}$ is the identity mapping of $\tilde{Y}$ into $\tilde{Y}$. Let $s = (u_0, \cdots, u_n)$ be an n-simplex of Y and let w be the barycenter of s. By a similar consideration as above, we can show that the set $\{\tilde{g}(u_i);\ i = 0, 1, \cdots, n\}$ forms an n-face s' of the simplex whose barycenter is $\tilde{g}(w)$. Since f is a simplicial mapping, we have $f(s') = s$. Obviously g is defined uniquely and is the required one.

References

[1] P. Alexandroff and H. Hopf: Topologie I, Berlin (1935).

[2] S. Eilenberg and N. E. Steenrod: Foundations of Algebraic Topology, Princeton (1952).

[3] S. Lefschetz: Algebraic Topology, Princeton (1942).