31. *Divergent Integrals as Viewed from the Theory of Functional Analysis. II*[*)]

By Tadashige ISHIHARA

(Comm. by K. KUNUGI, M.J.A., March 12, 1957)

§ 6. *The examination of analyticity.*

We can see after the integration by part that if $v(k, s)$ is an analytic function of k, v^* satisfies $\frac{\partial}{\partial k}v^*=0$ $\left(\frac{\partial}{\partial k}=\frac{1}{2}\left(\frac{\partial}{\partial\sigma}+i\frac{\partial}{\partial\tau}\right)\right)$, and $\varDelta v^*=0$ $\left(\varDelta=\frac{\partial^2}{\partial\sigma^2}+\frac{\partial^2}{\partial\tau^2}\right)$

However in our space $\varPhi'$ either the equation $\frac{\partial}{\partial k}v^*=0$ or $\varDelta v^*=0$ can not be a criterion of the analyticity of v^* unlikely to the case in $\mathfrak{D}'$. We see this fact easily from the following counter example. If $v\equiv 1$, both equations hold for v^*, but v^* is not regular at the origin (Example 2).

As already seen in § 3, no function $\varphi(\sigma, \tau)$ of $\varPhi$ has a compact carrier. However we saw also in § 3 that any element φ of $\mathfrak{D}_L(\sigma, \tau)$ can be approximated by $\{\varphi_j \mid \varphi_j \in \varPhi\}$ in the topology S. Hence we can see that when $v(k, s)$ is an analytic function of k, v^* is equivalent in $\varPhi'$ to an analytic function on a compact set $L(\subset D_1)$ if v^* is continuous for such sequence $\{\varphi_j \mid \varphi_j \xrightarrow[S]{} \varphi, \varphi \in \mathfrak{D}_L(\sigma, \tau), \varphi_j \in \varPhi\}$.

In the following we see three examples of our divergent integrals which are the Laplace transforms. Example 1 has no singularity on its abscissa of convergence. Example 2 has one singular point on its abscissa of convergence, and Example 3 has its natural boundary on its abscissa of convergence.

Example 1. $f(s)=\int_0^\infty e^{-st}F(t)dt$ where $F(t)=-\pi e^t \sin(\pi e^t)$. This integral diverges on $R(s)\leq 0$, and $\mathfrak{L}^{(k)}$-transform (by Cesàro's methods of summation of order k) is convergent on $R(s)>-k$ for arbitrary k [2].

We consider this integral as above, for example for the case $k=2$. We take the domain $-2+\varepsilon\leq\tau\leq\tau_2<\infty$, $-\infty<\sigma<+\infty$, as D_1. By repeated partial integration we see

$$f(s,t)=\int_0^t e^{-st}F(t)\,dt=1+e^{-st}\cos(\pi e^t)+\frac{s}{\pi}e^{-(s+1)t}\sin(\pi e^t)$$

$$-\frac{s(s+1)}{\pi^2}-\frac{s(s+1)}{\pi^2}e^{-(s+2)t}\cos(\pi e^t)$$

*) T. Ishihara [1].

$$-\frac{s(s+1)(s+2)}{\pi^2}\int_0^t e^{-(s+2)\tau}\cos(\pi e^{\tau})d\tau.$$

The 5th term of the right hand side converges to 0 as $t\to\infty$, and the 2nd and the 3rd terms diverge for $R(s)<0$.

Now putting $s=i(\sigma+i\tau)$ we consider the integral on $\Phi(\sigma,\tau)$.

$$\langle f(s),\varphi\rangle=\left\langle 1-\frac{s(s+1)}{\pi^2}-\frac{s(s+1)(s+2)}{\pi^2}\int_0^t e^{-(s+2)\tau}\cos(\pi e^{\tau})d\tau,\varphi\right\rangle$$
$$+\lim_{t\to\infty} v(t,\theta),$$

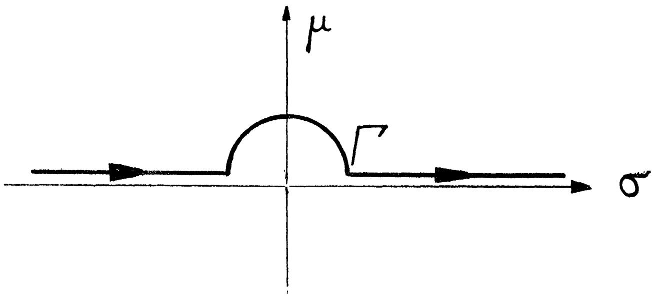
where

$$v(t,\theta)=\left\langle e^{-st}\cos(\pi e^{\theta})+\frac{s}{\pi}e^{-(s+1)t}\sin(\pi e^{\theta}),\varphi\right\rangle$$

We can see that there exists l such that $v(t,\theta)\in\mathfrak{D}_l(t)$ for all θ. So we see $\lim_{t\to\infty} v(t,t)=0$ and $f(s)$ is equal to the analytic function $1-\frac{s(s+1)}{\pi^2}-\frac{s(s+1)(s+2)}{\pi^2}\int_0^{\infty} e^{-(s+2)\tau}\cos(\pi e^{\tau})d\tau$ on $R(s)>-2$, on Φ. We can do similarly for arbitrary k and see that our integral $f(s)$ equals the analytic extension on the half plane $R(s)\leq 0$.

Example 2. We consider the case $v\equiv 1$, i.e. $f(s)=\int_0^{\infty} e^{iks}ds$, on D_1; $\tau_1<0<\tau_2$, $-\infty<\sigma<+\infty$. We see that

$$\langle f(s),\varphi(\sigma,\tau)\rangle=\int_0^{\infty}ds\int_{\Gamma}d\zeta\int_{\tau_1}^{\tau_2}e^{is\zeta}e^{-\tau s}\varphi(\zeta,\tau)d\tau.$$



Here $\zeta=\sigma+i\mu$ and the contour Γ is the curve shown in the above figure and $\varphi(\zeta,\tau)$ is the analytic extension of the function $\varphi(\sigma,\tau)$.

We can see that

$$\langle f,\varphi\rangle=\int_{\Gamma}d\zeta\int_{\tau_1}^{\tau_2}\frac{-1}{i(\zeta+i\tau)}\varphi(\zeta,\tau)d\tau$$
$$+\lim_{j\to\infty}\int\int\frac{e^{ij(\zeta+i\tau)}}{i(\zeta+i\tau)}\varphi(\zeta,\tau)d\zeta d\tau=\left\langle\frac{-1}{i(\zeta+i\tau)},\varphi\right\rangle$$

since the last term tends to 0 as in Example 1.

Now for any element φ of $\mathfrak{D}_L(\sigma,\tau)$ whose compact carrier L does not contain the origin, we take a sequence $\{\varphi_j\}$ of Φ which converges to φ in the topology $\mathcal{S}$.

Then we obtain

$$\lim_{j\to\infty}\left\langle \frac{-1}{i(\zeta+i\tau)}, \varphi_j\right\rangle = \lim\left\{\int_{\sigma_1}^{\sigma_2} \frac{-1}{i(\sigma+i\tau)}\varphi_j d\sigma + \varepsilon_j\right\} = \left\langle \frac{-1}{i(\sigma+i\tau)}, \varphi\right\rangle$$

This shows that v^* equals (as the element of $\mathcal{S}'$) the analytic function $-1/ik$ on L.

Example 3. The Laplace transform $f(s)=\int_0^\infty e^{-st}[\sqrt{t}]dt$, where [ ] means the integral part.

$f(s)$ equals $\frac{1}{s}\sum_{k=1}^\infty e^{-k^2 s}$, so it has natural boundary on $R(s)=0$. However even this divergent integral defines a functional on the half plane $R(s)\leq 0$ as the corollary of Theorem 3 shows.

§ 7. *Remarks.*

To investigate the analytic extension of $f(z)=\int_a^b f(\lambda, z)\,d\lambda$ mentioned in § 1, we have another way. That is to say, selecting suitable functional space $\Phi(\lambda)$, its element $\varphi(\lambda)$, $\Phi'(\lambda)$ and $T(\lambda, z)(\in\Phi'(\lambda))$ having z as a parameter, we rewrite as follows.

$$\int_a^b f(\lambda, z)d\lambda = \langle T(\lambda, z), \varphi(\lambda)\rangle.$$

The right hand side of this equation may often be defined and may be analytic on the larger domain than the left one. Especially if we can rewrite it, using $T(\lambda, z)$ such that the mapping from the complex plane to Φ', $z\to T(\lambda, z)$ is known to be weakly continuous for $z\in D_1\bigcup D$, we would have already obtained its analytic extension. The following example shows this case. Substantially this has no more than classical results, (for example, for the integral representation of Γ-function [3]), but we can see a functional theoretical expression of the analytic extension on the divergent domain.

Example 4. We consider the Mellin transform $f(\alpha)=\int_0^\infty z^{\alpha-1}\Phi(z)dz$. Here $\Phi(t)\in\mathcal{S}(t)$ for $0\leq t<\infty$. Generally the integral diverges on $R(\alpha)<0$. However we rewrite it by $f(\alpha)=\langle \text{p.f.}_{(z>0)}z^{\alpha-1}, \Phi(z)\rangle$. Then we can see $f(\alpha)$ can be analytically extended on the whole α-plane except $\alpha\neq -m$ (m is non-negative integer) and is expressed by

$$f(\alpha)=\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty \Phi(z)z^{\alpha-1}dz + \frac{\Phi(0)\varepsilon^z}{z} + \frac{\Phi'(0)\varepsilon^{z+1}}{z+1} + \cdots + \frac{\Phi^{(k)}(0)\varepsilon^{z+k}}{k!(z+k)}\right\}.$$

Especially in the case $\Phi(z)=e^{-z}$, we see an expression of Γ function on $R(\alpha)<0$.

References

[1] T. Ishihara: Divergent integrals as viewed from the theory of functional analysis. I, Proc. Japan Acad., **33**, 92–97 (1957).
[2] Doetsche: Handbuch der Laplace Transformation, **1**, 328, 151 (1952).
[3] L. Saalschutz: Bemerkungen über die Gammafunktion mit negativen Argumenten, Z. f. Math. u. Phy., **32**, 246–250 (1887).