45. *Some Examples of* (F) *and* (DF) *Spaces*

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(Comm. by K. KUNUGI, M.J.A., April 12, 1957)

In this paper we answer to the questions of A. Grothendieck [1] (question 1, 4, and 5 partly) giving some negative examples.

We say that an (F) space E has the *stability of the boundedness*, if for every bounded set B of E and every dense subspace E_0 of E, there exists a bounded set A of E_0 such that the closure of A includes B.

We show in §1 that there exists an (F) space which has not the stability of the boundedness. In §2, we give an example of reflexive (F) space in whose dual there exists a bounded set on which the strong topology is not metrizable. In §3, an example of bornologic (DF) space whose bidual is not bornologic is given.

§1. Let E be an (F) space, F a Banach space, and u a linear operator defined on a dense subspace E_0 of E into F, such that (1) $u(E_0)$ is dense in F, (2) $u^{-1}(0)$ is dense in E_0, and (3) for any bounded set B of E_0, the closure of $u(B)$ is not a neighbourhood of 0 in F. Then the graph of u, $G=\{(x, u(x)); x\in E_0\}$, in $E\times F$ is a dense subspace of $E\times F$, and the closure of any bounded set A of G does not include the unit sphere U of F considered as a subspace of $E\times F$. In fact, we have $A\subset B\times u(B)$ for the image B of A by the projection of $E\times F$ to E, while the closure of $B\times u(B)$ does not include U by the condition (3).

Thus $E\times F$ has not the stability of the boundedness, so we have only to give an example of such E, F and u.

Let S be a non-normable (F) space and B_λ $(\lambda\in\Lambda)$ a basis of bounded sets of S. Then we can find, for every $\lambda\in\Lambda$, a non-continuous linear functional $\varphi_\lambda\neq 0$ on S such that $\varphi_\lambda(x)=0$ for every $x\in B_\lambda$. Let E be $l^1(\Lambda, S)$, i.e. the usually defined (F) space of all functions $f(\lambda)$ of Λ into S such that $\sum_{\lambda\in\Lambda} f(\lambda)$ is absolutely summable in S. For $f\in E$, we define $u(f)$ as a function on Λ of which the value at λ is $\varphi_\lambda(f(\lambda))$. Now put $F=l^1(\Lambda)$, and $E_0=\{f; u(f)\in F\}$, then E_0 is dense in E and u, restricted on E_0, satisfies the conditions (1), (2) and (3).

In fact, $u(E_0)$ contains every function whose values are 0 except for a finite number of λ, and hence, is dense in F. The condition (2) is also satisfied, since $\varphi_\lambda^{-1}(0)$ is dense in S. Let A be an arbitrary bounded set of E_0, then there exists $\lambda_0\in\Lambda$ such that $f\in A$ implies $f(\lambda)\in B_{\lambda_0}$ for every $\lambda\in\Lambda$. Then the value of $u(f)$ at λ_0 is 0 for every $f\in A$, and hence the closure of $u(A)$ is not a neighbourhood of 0 in

F. Thus the condition (3) is satisfied.

§2. Let E be an (F) space and U_i $(i=1,2,\cdots)$ a basis of neighbourhoods of 0 in E. If the strong topology of E' is metrizable on the polar set U_i^0 of U_i for every $i=1,2,\cdots$, then there exists a double sequence of bounded sets $B_{i,j}$ of E such that $B_{i,j}^0 \frown U_i^0$ $(j=1,2,\cdots)$ constitutes a basis of neighbourhood of 0 in U_i^0 for the strong topology. Let B be a bounded set of E which absorbs every $B_{i,j}$, then B is *total*, that is, every non-zero element $x' \in E'$ is not constantly 0 on B. In fact, x' is in some U_i^0 and so not in some $B_{i,j}^0$. Therefore we have only to give an example of reflexive (F) space in which no bounded set is total.

Let Λ be the totality of monotone increasing sequence $\lambda=\{\xi_i\}$ $(\xi_i>0)$ and φ_i $(i=1,2,\cdots)$ functions on Λ defined as $\varphi_i(\lambda)=\xi_i$. Put $P_i(f)=(\sum_{\lambda\in\Lambda}|f(\lambda)|^2\varphi_i(\lambda))^{\frac{1}{2}}$ for every function f on Λ, and put $E=\{f;\ P_i(f)<+\infty$ for every $i=1,2,\cdots\}$. Then E is a reflexive (F) space with the semi-norms P_i. For an arbitrary bounded set B of E, put $\alpha_i=\sup_{f\in B} P_i(f)$, then we can find $\lambda_0=\{\xi_i\}$ so that $\lim_{i\to\infty}\xi_i^{-1}\alpha_i^2=0$. Since $|f(\lambda_0)|^2\xi_i\leqq P_i(f)^2\leqq\alpha_i^2$ for every $f\in B$ and $i=1,2,\cdots$, every f in B vanishes at λ_0 and hence B is not total.

§3. Let Λ be an infinite set and f_i $(i=1,2,\cdots)$ a sequence of positive functions on Λ such that $i<j$ implies $f_i\leqq f_j$. Then we obtain a bornologic (DF) space E as the inductive limit of Banach spaces E_i whose unit sphere is $\{f;\ |f(\lambda)|\leqq f_i(\lambda)\ (\lambda\in\Lambda)\}$. We shall show that E'' is not bornologic when functions f_i are adequately chosen.

Let F be the totality of functions g on Λ such that $\sum_{\lambda\in\Lambda}f_i(\lambda)|g(\lambda)|<+\infty$ for every $i=1,2,\cdots$, then F can be considered as a subspace of E' (as functionals $f\to\sum_{\lambda\in\Lambda}f(\lambda)g(\lambda)$) and also as the dual of a subspace E_0 of E which consists of all functions $f\in E$ whose values are 0 except for a finite number of λ. If $\varphi\in E'$ is continuous on E by the order-topology, then $\varphi\in F$. So there exists a continuous projection of E' onto F as well known in the theory of vector lattices. We can also obtain this same projection as the adjoint operator of the including mapping of E_0 into E.

Then there exists also a continuous projection of E'' onto F', and hence F' is a homomorphic image of E''.

Therefore, it is sufficient, for our purpose, to choose f_i as to make F' non-bornologic, that is, F "non-distingué". This case, however, has been already found in [1]—Λ is the set of pairs (n,m) of positive integers and

$$f_i(n,m)=\begin{cases} m & \text{if } n\leqq i \\ 1 & \text{if } n>i. \end{cases}$$

Reference

[1] A. Grothendieck: Sur les espaces (F) et (DF), Summa Brasiliensis Math., **3**, fasc. 6 (1954).