73. *Fourier Series. XVI. The Gibbs Phenomenon of Partial Sums and Cesàro Means of Fourier Series. 2*

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5. Proof of Theorem 7. Let

$$n_k = 2^{2^k} \quad (k=1, 2, \cdots).$$

Then $\quad 2\sqrt{n_k}\,\pi/n_k = 2\pi/\sqrt{n_k} = 2\pi/2^{2^{k-1}} = 2\pi/n_{k-1}.$

Let $\varphi_k(x)$ be an even concave function which is zero for $x \geqq \pi/2n_k$ and such that its curve touches y-axis at $y=1$ and touches x-axis at $x=\pi/2n_k$. Further suppose[1]

$$\int_0^t \varphi_k(x)\,dx - t\varphi_k(t) = t\Big/\sqrt{\log\log\frac{1}{t}}$$

for all $0 < t \leqq \pi/2n_k$.

Let

$$\begin{aligned} f_k(x) &= \varphi_k(x+(2j-1/2)\pi/n_k) \quad &&\text{in } ((2j-1)\pi/n_k,\ 2j\pi/n_k),\\ &= 0 &&\text{otherwise,} \end{aligned}$$

$$(j=\sqrt{n_k}/\log n_k,\ (\sqrt{n_k}/\log n_k)+1, \cdots, \sqrt{n_k}),$$

and

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Then

$$s_{n_k}(\pi/n_k, f) = s_{n_k}(\pi/n_k, f_k) + o(1).$$

If we set $\psi_k(t) = \varphi_k(t+\pi/2n_k)$, then

$$\begin{aligned} s_{n_k}(\pi/n_k, f_k) &= \frac{1}{\pi}\int_0^{\pi} f_k(t+\pi/n_k)\frac{\sin n_k t}{t}\,dt + o(1)\\ &= \frac{1}{\pi}\sum_{j=\sqrt{n_k}/\log n_k}^{\sqrt{n_k}} \int_0^{\pi/n_k} \psi_k(t)\frac{\sin n_k t}{t+2j\pi/n_k}\,dt + o(1)\\ &\geqq \frac{1}{\pi}\int_0^{\pi/n_k}\psi_k(t)\sin n_k t\,dt \sum_{j=\sqrt{n_k}/\log n_k}^{\sqrt{n_k}} \frac{n_k}{2j\pi} + o(1)\\ &\geqq A\log\log n_k\cdot n_k\int_0^{\pi/n_k}\psi_k(t)\sin n_k t\,dt + o(1)\\ &\geqq A\log\log n_k\cdot n_k\int_{\pi/4n_k}^{3\pi/4n_k}\psi_k(t)\,dt + o(1)\\ &\geqq A\log\log n_k/\sqrt{\log\log n_k}. \end{aligned}$$

Hence $s_{n_k}(\pi/n_k, f) \to \infty$ as $k \to \infty$. Thus partial sums of Fourier series of $f(x)$ present the Gibbs phenomenon at $x=0$.

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1) The base of logarithm is 2.

On the other hand, we can easily prove that

$$\int_0^t (f(u)+f(-u))\,du=o(t)$$

and

$$\int_0^t (f(x+u)-f(x-u))\,du=o(t)$$

uniformly for all x, and then by Theorem 5 Cesàro means do not present the Gibbs phenomenon. Thus Theorem 7 is proved.

By a slight modification of above example we can see that the condition (1) in Theorem 3 is best possible; that is, for any function $\omega(n)$ tending to infinity, however slowly may be, there is a function $f(x)$ such that

$$\int_0^h (f(x+u)-f(x-u))\,du=o\Big(h\omega(1/h)\Big/\log\frac{1}{h}\Big), \quad \text{uniformly in } x,$$

and partial sums of Fourier series of $f(t)$ present the Gibbs phenomenon at a certain point.

In Theorem 4 we can also say that the condition (1) is best possible. To see this we have to use $\varphi_k(x)$ modified such that its height tends to zero as $k\to\infty$.

6. Proof of Theorem 9. Let $0<r<1$ and let (m_k) and (n_k) be increasing sequences of integers, which will be determined later.

For a moment set $m_k=m$, $n_k=n$, $\alpha=1+r/2$ and $N=n+(1+r)/2$ Let β be an even integer determined later and we define the function $f_k(t)$ such that

(6) $$f_k(t)=-(t-\beta\pi/N)^{1+r}/((\alpha+\beta+1)\pi/N+2j\pi/N)^{1+r}$$

in the interval

$$((\alpha+\beta)\pi/N+2j\pi/N,\ (\alpha+\beta+1)\pi/N+2j\pi/N)$$

where $j=0,1,2,\cdots,m$ and otherwise $f_k(t)=0$. Then

(7) $$-1\leqq f_k(t)\leqq 0.$$

Using the notation in § 4, we write

$$\sigma_n^{(r)}(\beta\pi/N,f_k)=\frac{1}{\pi}\int_0^\pi f_k(t+\beta\pi/N)L_n^{(1)}(t)\,dt$$
$$+\frac{1}{\pi}\int_0^\pi f_k(t+\beta\pi/N)L_n^{(2)}(t)\,dt+\frac{1}{\pi}\int_0^\pi f_k(t+\beta\pi/N)L_n^{(3)}(t)dt$$
$$=I_1+I_2+I_3.$$

We have first

(8) $$I_1=\frac{1}{\pi A_n^r}\sum_{j=0}^m\int_{\alpha\pi/N}^{\alpha\pi/N+\pi/N} f_k\Big(t+\frac{\beta\pi}{N}+\frac{2j\pi}{N}\Big)\frac{\sin(Nt-r\pi/2)}{(t+2j\pi/N)^{1+r}}dt+o(1)$$
$$=\frac{1}{\pi A_n^r}\sum_{j=0}^m\frac{1}{((\alpha+\beta+1)\pi/N+2j\pi/N)^{1+r}}$$
$$\int_{\alpha\pi/N}^{\alpha\pi/N+\pi/N}(-\sin(Nt-r\pi/2))\,dt+o(1)$$

$$
\begin{aligned}
&=\frac{1}{\pi A_n^r}\left(\frac{N}{2\pi}\right)^{1+r}\frac{2}{N}\sum_{j=0}^{m}\frac{1}{((\alpha+\beta+1)/2+j)^{1+r}}+o(1)\\
&\geqq\frac{1+o(1)}{2^{1+r}\pi^{2+r}\Gamma(1+r)}\int_0^\infty\frac{dx}{(x+(\alpha+\beta+1)/2)^{1+r}}+o(1)\\
&=\frac{1+o(1)}{r\cdot 2\pi^{2+r}(\alpha+\beta+1)^r\Gamma(1+r)},
\end{aligned}
$$

if m is sufficiently large.

Secondly,

$$
\begin{aligned}
I_2&=-\frac{1}{\pi}\frac{r}{n+1}\sum_{j=0}^{m}\frac{1}{((\alpha+\beta+1)\pi/N+2j\pi/N)^{1+r}}\int_{\alpha\pi/N}^{\alpha\pi/N+\pi/N}\frac{dt}{(t+2j\pi/N)^{1-r}}\\
&\geqq-\frac{r(1+o(1))}{\pi^2}\sum_{j=0}^{m}\frac{1}{((\alpha+\beta+1)+2j)^{1+r}(\alpha+2j)^{1-r}}\\
&\geqq-\frac{r(1+o(1))}{4\pi^2}\left\{\int_0^\infty\frac{dx}{(x+\alpha/2)^{1-r}(x+(\alpha+\beta+1)/2)^{1+r}}+\frac{4}{\alpha^{1-r}(\alpha+\beta+1)^{1+r}}\right\}\\
&\geqq-\frac{r(1+o(1))}{2\pi^2}\left\{\frac{1}{\alpha}\int_0^\infty\frac{dx}{(x+1)^{1-r}(x+(\alpha+\beta+1)/\alpha)^{1+r}}+\frac{2}{\alpha^{1-r}(\alpha+\beta+1)^{1+r}}\right\}^{2)}\\
&\geqq-\frac{r(1+o(1))}{2\pi^2}\left\{\frac{1}{\alpha r(\beta+1)/\alpha}+\frac{2}{\alpha^{1-r}(\alpha+\beta+1)^{1+r}}\right\}\\
&=-\frac{1+o(1)}{2\pi^2(1+\beta)}-\frac{r(1+o(1))}{\pi^2\alpha^{1-r}(\alpha+\beta+1)^{1+r}}
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
|I_3|&\leqq\frac{8r(1-r)}{n^2}\sum_{j=0}^{m}\frac{1}{((\alpha+\beta+1)\pi/N+2j\pi/N)^{1+r}}\int_{\alpha\pi/N}^{\alpha\pi/N+\pi/N}\frac{dt}{(t+2j\pi/N)^{2-r}}\\
&\leqq\frac{8r(1-r)}{\pi^2}(1+o(1))\sum_{j=0}^{m}\frac{1}{((\alpha+\beta+1)+2j)^{1+r}(\alpha+2j)^{2-r}}\\
&\leqq\frac{r(1-r)}{\pi^2}(1+o(1))\left\{\frac{8}{\alpha^{2-r}(\alpha+\beta+1)^{1+r}}\right.\\
&\qquad\qquad\left.+\int_0^\infty\frac{dx}{(x+(\alpha+\beta+1)/2)^{1+r}(x+\alpha/2)^{2-r}}\right\}\\
&\leqq\frac{r(1-r)}{\pi^2}(1+o(1))\left\{\frac{8}{\alpha^{2-r}(\alpha+\beta+1)^{1+r}}+\frac{4}{(1-r)(\alpha+\beta+1)^{1+r}\alpha^{1-r}}\right\}\\
&\leqq\frac{12r(1+o(1))}{\pi^2}\frac{1}{\alpha^{1-r}(\alpha+\beta+1)^{1+r}}
\end{aligned}
$$

for large β.

Collecting above estimations we get

$$
\begin{aligned}
&\sigma_n^r(\beta\pi/N, f_k)\\
&\geqq\frac{1}{r\cdot 2\pi^{2+r}(\alpha+\beta+1)^r\Gamma(1+r)}-\frac{1}{2\pi^2(\beta+1)}-\frac{13r}{\pi^2\alpha^{1-r}(\alpha+\beta+1)^{1+r}}+o(1).
\end{aligned}
$$

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2) $\int_0^\infty\frac{dx}{(x+1)^{1-r}(x+q)^{1+r}}=\frac{1}{r(q-1)}\left[1-\frac{1}{q^r}\right]<\frac{1}{r(q-1)}$ $(q>1,\ 0<r<1)$.

The right side is greater than a positive constant g_r, if we take β suitably, depending only on r.

Let us suppose m_{k-1} and n_{k-1} are determined, then we take m_k and n_k such that (i) m_k is so large that the sum in (8) is sufficiently near to the infinite sum and (ii) $(\beta+2m_k)/n_k<1/n_{k-1}^2$. By such determined (m_k) and (n_k), we define $(f_k(x))$ and

$$f(x)=\sum_{k=1}^{\infty} f_k(x). \tag{9}$$

Then

$$\sigma_{n_k}^r(\beta\pi/N_k, f)=\sigma_{n_k}^r(\beta\pi/N_k, f_k)+o(1)\geqq g_r+o(1),$$

for all k. Thus, by (7), $\sigma_n^r(x, f)$ presents the Gibbs phenomenon at $x=0$.

We shall now prove Theorem 9. Let (ρ_k) be an increasing sequence tending to 1, and (r_k) be the sequence

$$r_1=\rho_1,\ r_2=\rho_1,\ r_3=\rho_2,\ r_4=\rho_1,\ r_5=\rho_2,\ r_6=\rho_3,\cdots,$$
$$r_{k(k+1)/2+1}=\rho_1,\ r_{k(k+1)/2+2}=\rho_2,\cdots,\ r_{(k+1)(k+2)/2}=\rho_k,\cdots.$$

In the definition (6) of $f_k(t)$, we use r_k instead of r, and let $f(t)=\sum f_k(t)$. Then this is the required function in Theorem 9.

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