72. Fourier Series. XVI. The Gibbs Phenomenon of Partial Sums and Cesàro Means of Fourier Series. 1

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1. There are many literatures concerning the Gibbs phenomenon of partial sums and Cesàro means of Fourier series of functions at jump points but a few concerning that at the points of discontinuity of the second kind (see B. Kuttner [1-4], O. Szász [5], S. Izumi and M. Satô [6] and K. Ishiguro [7, 8]). In our paper [6] we have proved

Theorem 1. *Suppose that*

$$f(t)=a\psi(t-\xi)+g(t)$$

where $\psi(t)$ *is a periodic function with period* 2π *such that*

$$\psi(t)=(\pi-t)/2 \qquad (0<t<2\pi)$$

and

$$-a\pi/2\leqq \liminf_{t\to\xi} f(t)\leqq \limsup_{t\to\xi} f(t)\leqq a\pi/2.$$

If

$$\int_0^t g(\xi+u)du=o(|t|),$$

and

$$\int_0^t (g(x+u)-g(x-u))du=o(|t|)$$

uniformly for all x *in a neighbourhood of* ξ, *then the Gibbs phenomenon of* $f(t)$ *appears at* $t=\xi$, *and the Gibbs set contains the interval* $[-a(H+1)\pi/4,\ a(H+1)\pi/4]$.

Theorem 2. *There is a function which does not present the Gibbs phenomenon at* $t=\xi$ *and has* $t=\xi$ *as the second kind discontinuity.*

We shall here prove

Theorem 3. *If*

$$(1)\qquad \int_0^h (f(x+u)-f(x-u))du=o\left(h\Big/\log\frac{1}{h}\right),\qquad \text{uniformly in } x,$$

then the partial sums of Fourier series of $f(t)$ *do not present the Gibbs phenomenon at all points.*

Using Theorem 3, we give a simple proof of Theorem 2. Further, as a particular case, we get the following theorem.

Theorem 4. *If* $f(t)$ *is continuous at a point* x *(or in an interval* (α,β) *or in* $(0,2\pi)$*), and (1) holds, then the Fourier series of* $f(t)$ *converges uniformly at* x *(or in a closed interval contained in* (α,β) *or in* $(0,2\pi)$*).*

This is a theorem of R. Salem [9] (the interval case) and one of us [10] (the point case), and proof of Theorem 3 gives a simple proof of Theorem 4.

Concerning Cesàro means

$$(2)\qquad \sigma_n^r(x,f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)K_n^r(t)\,dt,\ K_n^r(t)=\sum_{k=0}^{n}A_{n-k}^{r-1}D_k(t)/A_n^r,$$

we prove

Theorem 5. *If*

$$(3)\qquad \int_0^h(f(x+u)-f(x-u))du=o(h),\qquad \textit{uniformly in } x,$$

then Cesàro means of Fourier series of $f(t)$ of positive order do not present the Gibbs phenomenon at $t=0$.

From this we get the following theorem due to K. Ishiguro [8]:

Theorem 6. *If* $f(t)=a\psi(t)+g(t)$
where a is a constant and $g(t)$ satisfies the condition (3) in Theorem 5 and further

$$\limsup_{t\to0} f(t)\leqq a\pi/2,\qquad \liminf_{t\to0} f(t)\geqq -a\pi/2$$

then the Cesàro means $\sigma_n^r(x,f)$ of the Fourier series of $f(t)$ present the Gibbs phenomenon at $t=0$ for $r<r_0$ and not for $r\geqq r_0$, r_0 being the Cramér constant.

Theorem 7. *There is a function $f(t)$ such that the partial sums $s_n(x,f)$ present the Gibbs phenomenon, but not the Cesàro means $\sigma_n^r(x,f)$ for any positive order r.*

On the other hand, B. Kuttner [1] has proved

Theorem 8. *For any r $(0<r<1)$, there is a function $f(t)$ such that Cesàro means $\sigma_n^r(x,f)$ present the Gibbs phenomenon.*

His example is an unbounded function. We prove

Theorem 9. *There is a bounded function $f(t)$ such that the Cesàro means $\sigma_n^r(x,f)$ present the Gibbs phenomenon for any r, $0\leqq r<1$, at a point $x=0$.*

2. Proof of Theorem 3. We shall use the notations in [11]. Let $s_n(x,f)$ be the nth partial sum of Fourier series of $f(t)$ and let

$$\varphi_x(t)=f(x+t)+f(x-t).$$

Then

$$s_n(x,f)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\sin nt}{t}dt+o(1),$$

where the term $o(1)$ tends to zero uniformly as $n\to\infty$. After R. Salem (cf. [10]) we write

$$s_n(x,f)=\frac{1}{\pi}\sum_{k=0}^{n}(-1)^k\int_0^{\pi/n}\frac{\varphi_x(t+k\pi/n)}{t+k\pi/n}\sin nt\,dt+o(1)$$

$$=\frac{1}{\pi}\sum_{k=0}^{[n/2]}\int_0^{\pi/n}\left[\frac{\varphi_x(t+2k\pi/n)}{t+2k\pi/n}-\frac{\varphi_x(t+(2k+1)\pi/n)}{t+(2k+1)\pi/n}\right]\sin nt\,dt+o(1)$$

$$=\frac{1}{\pi}\sum_{k=0}^{[n/2]}\int_0^{\pi/n}\varphi_x(t+2k\pi/n)\left[\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k+1)\pi/n}\right]\sin nt\,dt$$

$$+\frac{1}{\pi}\sum_{k=0}^{[n/2]}\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k+1)\pi/n)}{t+(2k+1)\pi/n}\sin nt\,dt+o(1)$$

$$=I+J+o(1).$$

In order to prove the theorem it is sufficient to prove that $\liminf\limits_{x\to 0} f(x)\geqq 0$ implies

$$\liminf_{n\to\infty,\,x\to 0} s_n(x,f)\geqq 0. \tag{4}$$

We can suppose $f(x)\geqq 0$ by the local property of the partial sums. Then $I\geqq 0$ and hence it is sufficient to prove that $J=o(1)$.*) By the second mean value theorem, for $0\leqq\eta_k<\xi_k\leqq\pi/n$,

$$J=\frac{1}{\pi}\sum_{k=0}^{[n/2]}\frac{n}{(2k+1)\pi}\int_{\eta_k}^{\xi_k}[\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k+1)\pi/n)]dt$$

$$=o\left(\sum_{k=1}^{n}\frac{n}{k}\frac{1}{n\log n}\right)=o(1).$$

For, if we write

$$J'=\int_\eta^\xi[\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k+1)\pi/n)]dt$$

$$=\int_\eta^\xi[f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n)]dt$$

$$+\int_\eta^\xi[f(x-t-2k\pi/n)-f(x-t-(2k+1)\pi/n)]dt$$

$$=J_1'+J_2',$$

then

$$J'=\int_\eta^\xi[f(x+2k\pi/n+t)-f(x+2k\pi/n-t)]dt$$

$$-\int_\eta^\xi[f(x+(2k+1)\pi/n+t)-f(x+2k\pi/n-t)]dt$$

$$=\left[\int_0^\xi-\int_0^\eta\right][f(x+2k\pi/n+t)-f(x+2k\pi/n-t)]dt$$

$$-\left[\int_0^\xi-\int_0^\eta\right][f(x+(2k+1)\pi/n+t)-f(x+2k\pi/n-t)]dt,$$

where each integral is of order $o(1/n\log n)$ by the condition (1), uniformly in $k\leqq n$. Accordingly $J_1'=o(1/n\log n)$, and similarly $J_2'=o(1/n\log n)$; and hence $J'=o(1/n\log n)$ uniformly in $k\leqq n$.

3. Proof of Theorem 2. It is sufficient to prove that there is

*) Proof of $J=o(1)$ is the same as in [10].

a function satisfying the condition (1) and having a point of discontinuity of the second kind.

We can take an even function $f_k(x)$ such that

$$f_k(0)=1,\quad f_k(x)>0\quad (0\leqq x<b_k),\quad f_k(x)=0\quad (x\geqq b_k)$$

and

$$(5)\qquad \left|\int_0^t (f_k(x+u)-f_k(x-u))\,du\right|\leqq t\Big/\left(\log\frac{1}{t}\right)^2$$

for all t and x. For, if we take $f_k(x)$ such that the graph of $f_k(x)$ is concave in the interval $(0, b_k)$ and touches x-axis at $x=b_k$ and y-axis at $y=1$ and the integral of $f_k(x)$ on the interval $(0, t)$ $(0<t<b_k)$ minus $tf_k(t)$ is less than $1/2t(\log 1/t)^2$, then the condition (5) is satisfied.

Let (a_k) and (b_k) be sequences rapidly tending to zero such that

$$a_{k+1}<a_k/4,\quad 4b_k<a_k,\quad a_{k+1}+2b_{k+1}<a_k-2b_k,$$

and let

$$f(x)=\sum_{k=1}^{\infty}\{f_k(x+a_k+b_k)-f_k(x+a_k-b_k)\}.$$

Then

$$\int_0^t [f(x+u)-f(x-u)]\,du=o\left(t\Big/\log\frac{1}{t}\right),\qquad \text{as } t\to 0,$$

uniformly in x; and hence $f(x)$ has $x=0$ as a point of discontinuity of the second kind and the Gibbs phenomenon does not appear at $x=0$ by Theorem 3.

4. Proof of Theorem 5. Without any loss of generality we can suppose $f(0)=0$. As is well known [11, p. 184],

$$\sigma_n^r(x, f)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)K_n^r(t)\,dt$$

where

$$\begin{aligned}K_n^r(t)&=\frac{\sin[(n+(r+1)/2)t-r\pi/2]}{A_n^r(2\sin t/2)^{1+r}}+\frac{r}{n+1}\frac{1}{(2\sin t/2)^2}\\&\quad+\frac{1}{4A_n^r}\sum_{\nu=n+1}^{\infty}A_{\nu+1}^{r-2}\frac{\cos(\nu-n)t}{(2\sin t/2)^2}\\&=L_n^{(1)}(t)+L_n^{(2)}(t)+L_n^{(3)}(t),\end{aligned}$$

say. Putting $N=n+(r+1)/2$ and $\alpha=(2-r/2)\pi/N$,

$$\int_0^{\pi}\varphi_x(t)\,K_n^r(t)\,dt=\int_0^{\alpha}+\int_{\alpha}^{\pi}=I+J.$$

We have $I\geqq 0$. Let

$$\begin{aligned}J&=\int_{\alpha}^{\pi}\varphi_x(t)L_n^{(1)}(t)\,dt+\int_{\alpha}^{\pi}\varphi_x(t)L_n^{(2)}(t)\,dt+\int_{\alpha}^{\pi}\varphi_x(t)L_n^{(3)}(t)\,dt\\&=J_1+J_2+J_3,\end{aligned}$$

then

$$J_1=\int_\alpha^\pi \varphi_x(t)\frac{\sin(Nt-r\pi/2)}{A_n^r(2\sin t/2)^{1+r}}dt$$
$$=\frac{1}{A_n^r}\int_\alpha^\pi \frac{\varphi_x(t)}{t^{1+r}}\sin(Nt-r\pi/2)\,dt+o(1)$$
$$=\frac{1}{A_n^r}\sum_{k=1}^{[(N-1)/2]}\int_\alpha^{\alpha+\pi/N}\left\{\frac{1}{(t+2k\pi/N)^{1+r}}-\frac{1}{(t+(2k+1)\pi/N)^{1+r}}\right\}\cdot\varphi_x(t+2k\pi/N)\sin(Nt-r\pi/2)\,dt$$
$$+\frac{1}{A_n^r}\sum_{k=1}^{[(N-1)/2]}\int_\alpha^{\alpha+\pi/N}\frac{\varphi_x(t+2k\pi/N)-\varphi_x(t+(2k+1)\pi/N)}{(t+(2k+1)\pi/N)^{1+r}}\cdot\sin(Nt-r\pi/2)\,dt+o(1)$$
$$=J_{11}+J_{12}+o(1).$$

It is sufficient to prove that $f(t)\geqq 0$ implies $\sigma_n^r(x,f)\geqq o(1)$ where $o(1)$ is the term tending to zero as $n\to\infty$. Evidently $J_{11}\geqq 0$. By the second mean value theorem

$$|J_{12}|\leqq A\sum_{k=1}^{[(N-1)/2]}\frac{N}{k^{1+r}}\left|\int_{\xi_k}^{\eta_k}(\varphi_x(t+2k\pi/N)-\varphi_x(t+(2k+1)\pi/N))\,dt\right|$$

where $\alpha\leqq\xi_k<\eta_k\leqq\alpha+\pi/N$. By the condition (3), the right is $o(1)$, and then $J_1\geqq o(1)$. We have also $J_2\geqq 0$. It remains to prove that $J_3\geqq o(1)$. Now we put

$$J_3=\frac{1}{4A_n^r}\sum_{\nu=n+1}^{\infty}A_n^{r-2}\int_\alpha^\pi\frac{\varphi_x(t)\cos(\nu-n)t}{(2\sin t/2)^2}dt$$
$$=\frac{1}{4A_n^r}\left(\sum_{\nu=n+1}^{n+n_0}+\sum_{\nu=n+n_0+1}^{\infty}\right)=J_{31}+J_{32}$$

for an n_0. For a large but fixed n_0, $J_{31}\geqq o(1)$ and $J_{32}=o(1)$ by the condition (3), using the estimation as in J_1. Thus the theorem is proved.

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