88. On Dowker's Problem

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In 1949, C. H. Dowker raised the question[1] 'Is every normal Hausdorff space R countably paracompact (i.e. does every countable open covering of R have a locally finite open refinement)?'. In this paper, we shall give a negative answer to this problem, i.e. we shall show that there exists a normal Hausdorff space which is not countably paracompact.

(1) Let

$R_1=\{0, 1, 2, 3,\cdots, \omega,\cdots, \Omega\}$ where Ω is the first ordinal number in all 3rd-class ordinals,

$R_2=R_3=\cdots=R_n=\cdots=\{0, 1, 2, 3,\cdots, \omega\}$ where each $n<\infty$, ω is the first ordinal in all 2nd-class ordinals.

For each R_i, we define its topology by the limit of ordinals as usual.[2]

Let

$$S=R_1\times R_2\times R_3\times\cdots .$$

Give the weak topology of the product space for S.

Since each R_i is compact Hausdorff space, S is a compact Hausdorff space. And, therefore S is normal.

Now, $(\Omega, \omega, \omega, \omega,\cdots)$ is a point of S. Let

$$R=S-(\Omega, \omega, \omega, \omega,\cdots).$$

(2) Since R is a subspace of S, R is a Hausdorff space. We shall prove that R is normal.

Let A, B be disjoint two closed sets of R.

Let $\bar{A}$ be the closure in S of A, and $\bar{B}$ be the closure in S of B.

(i) The case of $\bar{A}\frown\bar{B}\not\ni(\Omega, \omega, \omega,\cdots)$.

$\bar{A}, \bar{B}$ are disjoint two closed sets of S. Since S is normal, there exist disjoint two open sets G_0, H_0 of S such that $G_0\supset\bar{A}$, $H_0\supset\bar{B}$. $G=R\frown G_0$, $H=R\frown H_0$ are disjoint two open sets of R such that $G\supset A$, $H\supset B$.

(ii) The case of $\bar{A}\frown\bar{B}\ni(\Omega, \omega, \omega,\cdots)$.

This case never happen.

Assume $\bar{A}\frown\bar{B}\ni(\Omega, \omega, \omega,\cdots)$.

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1) See [1]. (Numbers in brackets refer to the references at the end of the paper.)

2) We define neighbourhoods of p as follows; for each $q<p$, $\{p'\mid q<p'\leq p\}$ is a neighbourhood of p.

For any $\gamma<\Omega$, there exists a point $a\in A$ such that $a=(\alpha, \omega, \omega, \cdots)$ $(\gamma<\alpha<\Omega)$ by the following argument.

'Define open sets U_n $(n=1, 2, 3, \cdots<\infty)$ of S as follows:

$$U_n=\{p=(p_1, p_2, \cdots) \mid \gamma<p_1\leq\Omega,\ n-1\leq p_2\leq\omega,\ n-2\leq p_3\leq\omega, \cdots, 1\leq p_n\leq\omega,\ 0\leq p_i\leq\omega\ (i=n+1,\ n+2, \cdots)\},$$

then

$$\left.\begin{array}{l} U_n\frown A\neq\phi,^{3)} \\ U_n\frown B\neq\phi. \end{array}\right\} (n=1, 2, 3, \cdots)$$

Let $a_n\in U_n\frown A$, and let α_n be the first coordinate of a_n.

Consider the sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ of ordinals. Then there exists a subsequence $\{\alpha_{n_j} \mid j=1, 2, 3, \cdots<\infty\}$ such that $\alpha_{n_1}\leq\alpha_{n_2}\leq\alpha_{n_3}\leq\cdots$.

Let $\alpha=\lim\limits_{j\to\infty}\alpha_{n_j}$, then $\gamma<\alpha<\Omega$.

Let $a=(\alpha, \omega, \omega, \cdots)$, then a is an accumulation point of $\{a_n \mid n<\infty\}$ and $a\in R$. As A is a closed set of R, $a\in A$'.

By a similar argument, there exists a point $b\in B$ such that $b=(\beta, \omega, \omega, \cdots)$ $(\gamma<\beta<\Omega)$.

Now, let $a_1'\in A$, $a_1'=(\zeta_{a,1}, \omega, \omega, \cdots)$, $0<\zeta_{a,1}<\Omega$. There exists such a_1' by the above argument. As γ in the above argument is arbitrary, there exists a point $b_1'\in B$ such that $b_1'=(\zeta_{b,1}, \omega, \omega, \cdots)$, $\zeta_{a,1}<\zeta_{b,1}<\Omega$. And then there exists $a_2'\in A$ such that $a_2'=(\zeta_{a,2}, \omega, \omega, \cdots)$, $\zeta_{b,1}<\zeta_{a,2}<\Omega$. By similar arguments, we define $a_3', a_4', a_5', \cdots\in A$, $b_2', b_3', b_4', \cdots\in B$ one after another.[4] Both the sequence $\{a_i'\}$ and the sequence $\{b_i'\}$ converge to a same point $p=(\zeta, \omega, \omega, \cdots)(\zeta=\lim\limits_{n\to\infty}\zeta_{a,n}=\lim\limits_{n\to\infty}\zeta_{b,n}<\Omega)$ $\in R$. As both A and B are closed sets of R, $p\in A$ and $p\in B$. This result is contradictory to $A\frown B=\phi$.

By (i), (ii), R is normal.

(3) Define closed sets F_n $(n=1, 2, 3, \cdots<\infty)$ of R as follows:

$$F_1=\{p=(p_1, p_2, \cdots) \mid p_1=\Omega,\ 0\leq p_i\leq\omega\ (i=2, 3, 4, \cdots)\}-(\Omega, \omega, \omega, \cdots),$$

$$F_2=\{p=(p_1, p_2, \cdots) \mid p_1=\Omega,\ p_2=\omega,\ 0\leq p_i\leq\omega\ (i=3, 4, 5, \cdots)\}-(\Omega, \omega, \omega, \cdots),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$F_n=\{p=(p_1, p_2, \cdots) \mid p_1=\Omega,\ p_2=p_3=\cdots=p_n=\omega,\ 0\leq p_i\leq\omega\ (i=n+1, n+2, n+3, \cdots)\}-(\Omega, \omega, \omega, \cdots),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Then $F_1\supset F_2\supset F_3\supset\cdots$ is a decreasing sequence of closed sets of R with vacuous intersection.

We shall prove that there exists no sequence $\{G_n \mid n=1,2,3,\cdots<\infty\}$ of open sets of R satisfying the following conditions (Δ):

$$(\Delta)\qquad \begin{cases} G_n\supset F_n\ (n=1, 2, 3, \cdots), \\ \bigcap\limits_{n=1}^{\infty}\overline{G}_n=\phi \text{ where } \overline{G}_n \text{ is the closure in } R \text{ of } G_n. \end{cases}$$

3) We denote the empty set by ϕ.

4) We must use the mathematical induction in strictly.

Assume that there exists a sequence $\{G_n\}$ of open sets of R satisfying the conditions (Δ).

F_n contains

$$P_n=\{p=(p_1, p_2,\cdots) \mid p_1=\Omega,\ p_2=p_3=\cdots=p_n=\omega,\ p_{n+1}=p_{n+2}=\cdots=m,\ m=1, 2, 3,\cdots<\omega\}.$$

Therefore, for the open set $G_n \supset F_n$ there exist ordinals $\xi_{n,m}<\Omega$, $c_{n,m,2}<\omega$, $c_{n,m,3}<\omega,\cdots$, $c_{n,m,n}<\omega$; $(m=1, 2, 3,\cdots<\omega)$ such that

$$\begin{cases} Q_{n,m}=\{p=(p_1, p_2,\cdots) \mid \xi_{n,m}<p_1\leq\Omega,\ c_{n,m,2}<p_2\leq\omega,\ c_{n,m,3}<p_3\leq\omega,\cdots, \\ \qquad c_{n,m,n}<p_n\leq\omega,\ p_{n+1}=p_{n+2}=\cdots=m\}, \\ \bigcup_{m=1}^{\infty} Q_{n,m}\subset G_n. \end{cases}$$

Let $\xi_n=\sup_m\{\xi_{n,m}\}$, and let

$$T_n=\{p=(p_1, p_2,\cdots) \mid \xi_n<p_1<\Omega,\ p_2=p_3=\cdots=\omega\},$$

then $T_n\subset \overline{G}_n$ by the following argument.

'Let $t_n \in T_n$. Then

$$t_n=(\tau_n, \omega, \omega,\cdots)\ (\xi_n<\tau_n<\Omega).$$

For each $m<\omega$ there exists $t_{n,m}$ such that

$$\begin{cases} t_{n,m}\in Q'_{n,m}=\{p=(p_1, p_2,\cdots) \mid \xi_{n,m}<p_1\leq\Omega,\ p_2=p_3=\cdots=p_n=\omega, \\ \qquad p_{n+1}=p_{n+2}=\cdots=m\}, \\ \text{the first coordinate of } t_{n,m} \text{ is } \tau_n. \end{cases}$$

Consider the sequence

$$t_{n,1},\ t_{n,2},\ t_{n,3},\ \cdots\ .$$

This sequence converges to t_n. For each m,

$$t_{n,m}\in Q'_{n,m}\subset Q_{n,m}\subset G_n.$$

Therefore $t_n\in\overline{G}_n$. Hence we have $T_n\subset\overline{G}_n$'.

Let $\xi=\sup_n\{\xi_n\}$, and let

$$T=\{p=(p_1, p_2,\cdots) \mid \xi<p_1<\Omega,\ p_2=p_3=\cdots=\omega\},$$

then $T_n\supset T$ for every integer n.

Therefore $\bigcap_{n=1}^{\infty}\overline{G}_n\supset\bigcap_{n=1}^{\infty}T_n\supset T$.

As T is non-empty, $\bigcap_{n=1}^{\infty}\overline{G}_n$ is non-empty. This result is contradictory to the assumption $\bigcap_{n=1}^{\infty}\overline{G}_n=\phi$.

Therefore, there exists no sequence $\{G_n\}$ of open sets of R satisfying the conditions (Δ).

(4) We shall prove that R is not countably paracompact.

From (2), R is a normal Hausdorff space. And, for the decreasing sequence $F_1\supset F_2\supset F_3\supset\cdots\to\phi$ of closed sets of R which is defined in (3), from the argument in (3) there exists no decreasing sequence $G_1\supset G_2\supset G_3\supset\cdots$ of open sets of R such that

$$\begin{cases} G_i \supset F_i \ (i=1, 2, 3, \cdots), \\ \bigcap_{i=1}^{\infty} \overline{G}_i = \phi. \end{cases}$$

Therefore, by the result in F. Ishikawa's paper [2], R is not countably paracompact.

Thus we conclude that R is a normal Hausdorff space which is not countably paracompact.

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References

[1] C. H. Dowker: On countably paracompact spaces, Canadian Jour. Math., **3**, 219–224 (1951).

[2] F. Ishikawa: On countably paracompact spaces, Proc. Japan Acad., **31**, 686–687 (1955).