124. *Non-Connection Methods for the Theory of Principal Fibre Bundles as Almost Kleinean Geometries*

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In some of the previous papers of the present author (T. Takasu [8, 9]), the author has established, on one hand, six non-holonomic geometries as double geometries consisting of the respective connection geometries and the respective non-holonomic geometries referred to the connection parameters of the teleparallelisms (É. Cartan [1]), having discovered the most remarkable fact that *the paths of the teleparallelisms* (i.e. the II-geodesic curves) *behave as for meet and join like straight lines,* what has led the author to the discovery of the actual and final formulation of the general theory of relativity as the 3-dimensional Laguerre principal fibre bundle geometry (T. Takasu [11]).

S. S. Chern and C. Ehresmann (S. S. Chern [2]; C. Ehresmann [3, 4]; A. Lichnerowicz [5]; K. Nomizu [6]; T. Ohtsuki [7]) established, on the other hand, a theory of connections as that of the cross sections of the (principal) fibre bundles introducing connections into them.

In this note, it will be shown firstly that *the present author's theory of the respective principal fibre bundles* (*based on the* II-*geodesic curves*) *is substantially nothing other than the respective theory in the large of S. S. Chern or that of C. Ehresmann,* since the II-geodesic curves do actually exist in the differentiable manifolds in the sense of them. Indeed É. Cartan [1] has once declared: "Les connexions affines que j'ai introduites rentrent dans les connexions encore plus générales dues à M. Schouten (Math. Zeitschr., 13, 56–81 (1922)); mais le point de vue de M. Schouten est différent du mien. Pour lui le transport parallèle (lineare Übertragung) est la notion géométrique essentielle; *pour moi, elle n'est qu'un moyen qui tient aux propriétés de l'espace affine et qui ne peut plus s'utiliser, au moins directement, pour établir la notion d'espace à connexion projective (ou conforme, etc.)" and the present author has the same notion as É. Cartan had, for, the choice of a connection for one and the same differentiable manifold corresponding to a Lie group means a choice of the paths as tangents to given curves and given subvarieties.*

It will also be shown secondly that the present author's theory of the respective principal fibre bundles based on the II-geodesic curves provides us *non-connection methods for the differentiable manifolds admitting infinitely many connections* and that *the results reduce to such an extent that the geometries under consideration become the cor-*

responding almost Kleinean geometries ("Erlanger Programm"), thus proving to form the *primary* ("haupt") *half* of the geometries under consideration, while the results of the classical connections form the secondary ("neben") half.

For shortness' sake, the case of the Euclidean principal fibre bundle will be treated. But the general procedure is, mutatis mutandis, common to the affine connection, the Weyl's equiform connection, the Veblen's non-Euclidean connection (O. Veblen [12]), the Möbius connection (T. Ohtsuki [7]), the projective connection (T. Ohtsuki [7]), the Laguerre connection (T. Takasu [9, 10]), the Lie connection (T. Takasu [9]), and the parabolic Lie connection (T. Takasu [9]).

In the Euclidean principal fibre bundle geometry, *the Lie group reduces to an orthogonal transformation group with position functions as coefficients, when* II-*geodesic analogues to the rectangular Cartesian coordinates are adopted. The*

II-*geodesic*	*geodesic*

curves are the solutions of the extremal problem $\delta s=0$ *in the*

Euclidean principal fibre bundle	*Riemannian*

geometry. Trigonometry and Hesse's normal form for II-geodesic $(n-1)$-flat V^{n-1} are introduced.

Considerable contributions for algebraic geometry, topology and function theory of many variables are expected.

1. Differentiable manifolds. See T. Takasu [9, Art. 1].

2. II-Geodesic curves. Adopting hypercomplex units γ_l such that

$$\gamma_m\gamma_n+\gamma_n\gamma_m=2\delta_{mn}, \quad (l, m, n, \cdots=1, 2, \cdots, n), \tag{2.1}$$

we put

$$dS=\gamma_l\omega^l, \quad \omega^l=\omega^l_\mu(x^\lambda)dx^\mu, \quad (\lambda, \mu, \cdots=1, 2, \cdots, n), \tag{2.2}$$

where it is assumed that

$$|\omega^l_\mu| \neq 0. \tag{2.3}$$

Then we have

$$dSdS=ds^2=\omega^l\omega^l=g_{\mu\nu}dx^\mu dx^\nu, \quad ds=|dS|, \tag{2.4}$$

$$g_{\mu\nu}=g_{\underline{\mu\nu}}+g_{\underset{\vee}{\mu\nu}}, \; g_{\underline{\mu\nu}}=\omega^l_\mu\omega^l_\nu=g_{\underline{\nu\mu}}, \; g_{\underset{\vee}{\mu\nu}}=\gamma_l\gamma_m\omega^l_\mu\wedge\omega^m_\nu=-g_{\underset{\vee}{\nu\mu}}, \tag{2.5}$$

$$g_{\underline{\mu\nu}}=\omega^l_\mu\omega^l_\nu, \quad g^{\underline{\mu\nu}}=\Omega^\mu_l\Omega^\nu_l, \tag{2.6}$$

where

$$\Omega^\lambda_l\omega^l_\mu=\delta^\lambda_\mu, \quad \Omega^\lambda_n\omega^m_\lambda=\delta^m_n. \tag{2.7}$$

Since the ω^l are written in invariant forms, they are *global* in $\bigcup_\alpha U_\alpha$ considered in Art. 1.

The Lie group consists in (i) the orthogonal transformations of ω^l with function coefficients, under which $ds^2=\omega^l\omega^l$ is invariant, and (ii) the coordinate transformations $\bar{x}^\lambda=\bar{x}^\lambda(x^\nu)$ either in one and the same open subset U_α or in $U_\alpha \cap U_\beta \neq \phi$ (satisfying the so-called paste conditions).

The identity

$$(2.8)\quad \frac{d}{ds}\frac{\omega^l}{ds}\equiv\omega^l_\lambda\left(\frac{d}{ds}\frac{\Omega^\lambda}{ds}+\Lambda^\lambda_{\mu\nu}\frac{\Omega^\mu}{ds}\frac{\Omega^\nu}{ds}\right) \quad\Big|\quad \frac{d}{ds}\frac{\Omega^\lambda}{ds}\equiv\Omega^\lambda_l\left(\frac{d}{ds}\frac{\omega^l}{ds}+\Lambda^l_{mn}\frac{\omega^m}{ds}\frac{\omega^n}{ds}\right)$$

is readily shown, where

$$(2.9)\quad \Omega^\lambda=\Omega^\lambda_l(x^\nu)\omega^l=dx^\lambda,$$

$$(2.10)\quad \Lambda^\lambda_{\mu\nu}\equiv\Omega^\lambda_l\frac{\partial\omega^l_\mu}{\partial x^\nu}\equiv-\omega^l_\mu\frac{\partial\Omega^\lambda_l}{\partial x^\nu}. \quad\Big|\quad \Lambda^l_{mn}=\omega^l_\lambda\frac{\partial\Omega^\lambda_m}{\omega^n}\equiv-\Omega^\lambda_m\frac{\partial\omega^l_\lambda}{\omega^n}.$$

Also later by (2.13) we shall have

$$(2.11)\quad \frac{dx^\lambda}{ds}=a^l\Omega^\lambda_l.$$

The $\Lambda^\lambda_{\mu\nu}$ is known as the parameter of *teleparallelism*.

The solution of the extremal problem

$$\delta s=0$$

in the principal fibre bundle $(\omega^l_\mu, x^\lambda)$ is

$$(2.12)\quad \frac{d}{ds}\frac{\omega^l}{ds}=0.$$

in the manifold $\{x^\lambda\}$ is

$$\frac{d^2x^\lambda}{ds^2}+\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=0.$$

The left-hand side gives

$$(2.13)\quad \xi^l\equiv\int\frac{\omega^l}{ds}ds=a^ls+c^l,\quad (a^la^l=1),$$

whose domain of existence being $U=\bigcup_\alpha U_\alpha$. We call the curves (2.13) the II-*geodesic curves* (the geodesic curves of the second kind).

The finite equations (2.13) show that *the* II-*geodesic curves behave as for meet and join like straight lines* (cf. (5.5)).

By means of the II-*geodesic curves, our differentiable manifolds can be mapped either into or onto the Euclidean space* E^n.

Illustrative example. The spherical surface is mapped by this principle *into* the Euclidean plane giving rise to the so-called Mercator chart.

3. The Euclidean principal fibre bundle based on the II-geodesic curves. Since the II-geodesic curves are, unlike the geodesic curves, free from singularity and *exist in the large in* $\bigcup_\alpha U_\alpha$, it is extremely profitable *to adopt a rectangular coordinate system* (ξ^l) *with* II-*geodesic curves as axes* replacing x^λ by $\delta^\lambda_l\xi^l$. Then the formula $d\bar{\xi}^l=\omega^l_\mu(x^\lambda)dx^\mu$ becomes

$$(3.1)\quad d\bar{\xi}^l=a^l_m(\xi^n)d\xi^m,$$

both (ξ^l) and $(\bar{\xi}^l)$ being II-*geodesic rectangular coordinates*, where *the matrix* $(a^l_m(\xi^n))$ *is an orthogonal one* satisfying ${}_nC_2+n$ conditions.

The integral of (3.1) becomes

$$(3.2)\quad \bar{\xi}^l=\int a^l_m(\xi^n)d\xi^m=a^l_m(\xi^n)\xi^m-\int\xi^m da^l_m(\xi^n),$$

which is of the form

$$\bar{\xi}^l = a^l_m(\xi^n)\xi^m + a^l_0(\xi^n), \tag{3.3}$$

the $a^l_0(\xi^n)$ being determined by $a^l_m(\xi^n)$ except the additive constants. The essential number of parameters a^l_k is $n^2-\frac{1}{2}n(n+1)=\frac{1}{2}n(n-1)$. *The manifold* $(a^l_m(\xi^n), a^l_0(\xi^n))$ *makes the Lie group,* to which the (i) and the (ii) of Art. 2 reduce. *Both sides of* (2.8) *and* (2.10) *coincide.*

Our *principal fibre bundle consists of the differentiable manifold* (ξ^l) *together with the Lie group* $(a^l_m(\xi^n), a^l_0(\xi^n))$; its dimension is $n^2-{}_nC_2-n+n=\frac{1}{2}n(n+1)$. *The* ds^2 *is an invariant under the Lie group.*

Thus *our global differential geometry of the principal fibre bundle based on the* II-*geodesic curves is nothing other than that of S. S. Chern and C. Ehresmann, provided that after É. Cartan* (cf. Introduction) *the connections are considered to be mere means* ("moyen").

4. The equations of structure. The equations of structure (cf. T. Takasu [9])

	Global:	Local:
(4.1)	$\boldsymbol{d}\omega^l-\omega^k\wedge\theta^l_k=\frac{1}{2}T^l_{mn}\omega^m\wedge\omega^n,$ $\boldsymbol{d}\theta^k_l-\theta^t_l\wedge\theta^k_t=\frac{1}{2}R^k_{\cdot lmn}\omega^m\wedge\omega^n$	$\mathrm{d}\Omega^\lambda-\Omega^\kappa\wedge\theta^\lambda_\kappa=\frac{1}{2}T^\lambda_{\mu\nu}\Omega^\mu\wedge\Omega^\nu,$ $\mathrm{d}\theta^\kappa_\lambda-\theta^\tau_\lambda\wedge\theta^\kappa_\tau=\frac{1}{2}R^\kappa_{\cdot\lambda\mu\nu}\Omega^\mu\wedge\Omega^\nu$

reduce to

(4.2)	$\boldsymbol{d}\omega^l=0$	$\mathrm{d}\Omega^\lambda=0$

owing to

$\theta^k_l\equiv 0,\ T^l_{mn}\equiv 0,\ R^k_{\cdot lmn}\equiv 0,$	$\theta^\kappa_\lambda\equiv 0,\ T^\lambda_{\mu\nu}\equiv 0,\ R^\kappa_{\cdot\lambda\mu\nu}\equiv 0,$

when the II-geodesic rectangular coordinates (ξ^l) are adopted in place of (x^λ). The condition (4.2) turns out into

(4.3)	$\Lambda^\lambda_{\mu\nu}\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}=0.$	$\Lambda^l_{mn}\dfrac{\omega^m}{ds}\dfrac{\omega^n}{ds}=0.$

Consequently we have

(4.4)	$\dfrac{d}{ds}\dfrac{\Omega^\lambda}{ds}\equiv\dfrac{d^2x^\lambda}{ds^2}+\Lambda^\lambda_{\mu\nu}\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds}=0,$	$\dfrac{d}{ds}\dfrac{\omega^l}{ds}\equiv\dfrac{d}{ds}\dfrac{\omega^l}{ds}+\Lambda^l_{mn}\dfrac{\omega^m}{ds}\dfrac{\omega^n}{ds}=0,$

both sides of (4.1), (4.2), (4.3) and (4.4) coinciding. Thus *the* II-*geodesic curves* (2.12) *arise also as special ones of the paths* (4.4), *and the equations of structure turn out into those of the* II-*geodesic curves.*

5. Trigonometry and polar coordinates. Introduce a new system $(\overset{p}{\underset{s}{e}})$ of hypercomplex units such that

$$\overset{p}{\underset{s}{e}}=\overset{p}{\gamma}_1\overset{p}{\gamma}_s,\quad (\overset{p}{\underset{1}{e}}=1),\quad (p, q, s, \cdots=1, 2, \cdots, n), \tag{5.1}$$

where $(\overset{p}{\gamma}_s)$ is the conjugate set of the hypercomplex units (γ_l) of (2.1).

We define the generalized n-metric cosines $\underset{q}{C}(\underset{s}{\theta})$ by the formulas

$$e^{\left(\overset{p}{\underset{s}{e}}-\frac{1}{n}\underset{s}{\alpha}\right)\underset{s}{\theta}}=\underset{q}{C}(\underset{s}{\theta})\overset{p}{\underset{q}{e}},\quad \left(\overset{p}{\underset{1}{e}}=1;\ \frac{1}{n}\underset{s}{\alpha}=\sum_{p=1}^{n}\overset{p}{\underset{s}{e}}\right). \tag{5.2}$$

Solving (5.2) by Cramer's method, we obtain

$$E\cdot\underset{q}{C} = \overset{q}{\underset{p}{E}}\, e^{(\overset{p}{\underset{s}{e}} - \frac{1}{n}\underset{s}{\alpha})\underset{s}{\theta}}, \tag{5.3}$$

where

$$\delta_q^{q'} E \equiv \delta_q^{q'} |\overset{p}{\underset{s}{e}}| = \overset{p}{\underset{q}{e}}\, \overset{q'}{\underset{p}{E}}, \quad \delta_p^{p'} E \equiv \delta_p^{p'} |\overset{p}{\underset{s}{e}}| = \delta_q^{p'} \overset{q}{\underset{p}{E}}. \tag{5.4}$$

As for the meaning of $\underset{s}{\theta}$, see T. Takasu [13].

We may put (cf. (2.13))

$$\xi^l = \rho\cdot \underset{l}{C}(\underset{s}{\theta}), \tag{5.5}$$

for,

$$\prod_{p=1}^{n} (\overset{p}{\underset{l}{e}}\, \xi^l) = \rho^n \prod_{p=1}^{n} \overset{p}{\underset{q}{e}}\, \underset{q}{C}(\underset{s}{\theta}) = \rho^n \prod_{p=1}^{n} e^{(\overset{p}{\underset{s}{e}} - \frac{1}{n}\underset{s}{\alpha})\underset{s}{\theta}} = \rho^n, \tag{5.6}$$

where the ρ is the II-*geodesic modulus*. For $\rho=1$, we see that

$$(\overset{1}{\underset{l}{e}}\, \underset{l}{C}(\underset{s}{\theta}))^2 = (\gamma_1)^2(\gamma_l \underset{l}{C}(\underset{s}{\theta}))^2 = \underset{l}{C}(\underset{s}{\theta})\, \underset{l}{C}(\underset{s}{\theta}) = \rho^2 = 1,$$

$$\underset{l}{C}(\underset{s}{\theta})\, \underset{l}{C}(\underset{s}{\theta}) = 1. \tag{5.7}$$

Hesse's normal form for II-geodesic $(n-1)$-flat V^{n-1} may be introduced.

6. A theory of curves with II-geodesic curves as tangents

The left-hand side of the following lines gives a theory of curves with II-*geodesic curves as tangents* and coincides *in form* with that of the Euclidean space E^n.

Global (principal fibre bundle)	Local (Riemannian)
$\xi^l = \xi^l(s)$	$x^\lambda = x^\lambda(s)$
$\overset{1}{\xi^l} \equiv \dfrac{d\xi^l}{ds}$	$\overset{1}{\xi^\lambda} \equiv \dfrac{dx^\lambda}{ds}$
$\overset{1}{\xi^l}\overset{1}{\xi^l} = 1, \quad \xi^l \dfrac{d\overset{1}{\xi^l}}{ds} = 0$	$g_{\mu\nu}\overset{1}{\xi^\mu}\overset{1}{\xi^\nu} = 1, \quad g_{\mu\nu}\overset{1}{\xi^\mu}\dfrac{\delta\overset{1}{\xi^\nu}}{\delta s} = 0$
$(\overset{1}{\kappa})^2 = \dfrac{d\overset{1}{\xi^l}}{ds}\dfrac{d\overset{1}{\xi^l}}{ds}$	$(\overset{1}{\kappa})^2 = \dfrac{\delta\overset{1}{\xi^\lambda}}{\delta s}\dfrac{\delta\overset{1}{\xi^\lambda}}{\delta s}$
$\dfrac{d}{ds}\overset{1}{\xi^l} \equiv \overset{1}{\kappa}\overset{2}{\xi^l}$ etc.	$\dfrac{\delta}{\delta s}\overset{1}{\xi^\lambda} \equiv \overset{1}{\kappa}\overset{2}{\xi^\lambda}$ etc.

Frenet-Serret formulas

$\dfrac{d}{ds}\overset{\alpha}{\xi^l} = -\overset{\alpha-1}{\kappa}\overset{\alpha-1}{\xi^l} + \overset{\alpha}{\kappa}\overset{\alpha+1}{\xi^l}, \quad \overset{0}{\kappa} = \overset{n}{\kappa} = 0$	$\dfrac{\delta}{\delta s}\overset{\alpha}{\xi^\lambda} = -\overset{\alpha-1}{\kappa}\overset{\alpha-1}{\xi^\lambda} + \overset{\alpha}{\kappa}\overset{\alpha+1}{\xi^\lambda}, \quad \overset{0}{\kappa} = \overset{n}{\kappa} = 0$
$(\alpha = 1, 2, \cdots, n)$	$(\alpha = 1, 2, \cdots, n)$

7. V^m immersed in V^n.

The first column of the following lines shows *the theory of V^m immersed in V^n with* II-*geodesic curves as tangents to V^m*.

Global V^m, global V^n	Global V^m, local V^n	Local V^m, global V^n	Local V^m, local V^n
$\xi^p = \xi^p(\xi^1, \cdots, \xi^m)$	$x^\lambda = x^\lambda(\xi^1, \cdots, \xi^m)$	$\xi^p = \xi^p(u^1, \cdots, u^m)$	$x^\lambda = x^\lambda(u^1, \cdots, u^m)$
$g_{\mu\nu} \equiv g_{pq} = \delta_{pq}$	$g_{\mu\nu}$	$g_{\mu\nu} \equiv g_{pq} = \delta_{pq}$	$g_{\mu\nu}$
$g_{ij} = \delta_{ij}$	$g_{ij} = \delta_{ij}$	g_{ij}	g_{ij}
0	$\left\{ {\lambda \atop \mu\nu} \right\}$	0	$\left\{ {\lambda \atop \mu\nu} \right\}$

0	0	$\left\{ {i \atop jk} \right\}$	$\left\{ {i \atop jk} \right\}$
0	$R^{\lambda}_{\cdot\mu\nu\tau}$	0	$R^{\lambda}_{\cdot\mu\nu\tau}$
0	0	$R^{i}_{\cdot jhk}$	$R^{i}_{\cdot jhk}$
$B_j^{\cdot p}=\dfrac{\partial\xi^p}{\partial\xi^j}$	$B_j^{\cdot\mu}=\dfrac{\partial x^\mu}{\partial u^j}$	$B_j^{\cdot p}=\dfrac{\partial\xi^p}{\partial\xi^j}$	$B_j^{\cdot\mu}=\dfrac{\partial x^\mu}{\partial u^j}$
$\bar\xi^p=\bar\xi^p(\xi^1,\cdots,\xi^n)$ $=\xi^q\omega^p_q-\int\xi^q d\omega^p_q,$ $(d\bar\xi^p=\omega^p_q d\xi^q)$	$\bar x^\lambda=\bar x^\lambda(x^1,\cdots,x^n),$	$\bar\xi^p=\bar\xi^p(\xi^1,\cdots,\xi^n)$ $=\xi^q\omega^p_q-\int\xi^q d\omega^p_q,$ $(d\bar\xi^p=\omega^p_q d\xi^q)$	$\bar x^\lambda=\bar x^\lambda(x^1,\cdots,x^n)$
$\bar\xi^i=\bar\xi^i(\xi^1,\cdots,\xi^m)=\xi^j\omega^i_j-\int\xi^j d\omega^i_j,$ $(d\bar\xi^i=\omega^i_j d\xi^j)$		$\bar u^i=\bar u^i(u^1,\cdots,u^m)$	
etc.	etc.	etc.	etc.
$\dfrac{d^2\xi^p}{ds^2}=B_i^{\cdot p}\dfrac{d^2\xi^i}{ds^2}+H_{ij}^{\cdot\cdot p}\dfrac{d\xi^i}{ds}\dfrac{d\xi^j}{ds}$	$\dfrac{\delta}{\delta s}\dfrac{dx^\lambda}{ds}=B_i^{\cdot\lambda}\dfrac{d^2\xi^i}{ds^2}+H_{ij}^{\cdot\cdot\lambda}\dfrac{d\xi^i}{ds}\dfrac{d\xi^j}{ds}$	$\dfrac{\delta}{\delta s}\dfrac{d\xi^p}{ds}=B_i^{\cdot p}\dfrac{\delta}{\delta s}\dfrac{d\xi^i}{ds}+H_{ij}^{\cdot\cdot p}\dfrac{d\xi^i}{ds}\dfrac{d\xi^j}{ds}$	$\dfrac{\delta}{\delta s}\dfrac{dx^\lambda}{ds}=B_i^{\cdot\lambda}\dfrac{\delta}{\delta s}\dfrac{du^i}{ds}+H_{ij}^{\cdot\cdot p}\dfrac{du^i}{\delta s}\dfrac{du^j}{ds}$

(Absolute curvature vector=relative curvature vector+normal curvature vector)

The mean curvature vector, umbilics, asymptotic curves, lines of curvature, Weingarten's equations, Gauss' equations, Codazzi-Mainardi equations, Gaussens Theorema egregium, fundamental theorem of V^m-theory, theory of harmonic integrals, etc. may similarly be examined.

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