3. Note on Finite Simple c-Indecomposable Semigroups

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In this note we shall report the result of study of finite simple c-indecomposable semigroups except groups without proof, which we shall discuss precisely in another paper. A semigroup is said to be *c-indecomposable* if it has no commutative homomorphic image except one-element semigroup.

1. Finite simple semigroups. A simple semigroup is defined as a semigroup which has no proper ideal.[1]

Referring Theorem 8 in [1],[2] we have

Lemma 1. *A finite simple semigroup without zero belongs to one of the following three categories.*

(1) *Finite simple c-indecomposable semigroups without zero except groups.*

(2) *Finite groups.*

(3) *Finite simple non-commutative non-unipotent semigroups whose greatest c-homomorphic images are non-trivial groups.*

Lemma 2. *A finite simple semigroup with zero belongs to one of the following three categories.*

(1) *Finite simple c-indecomposable semigroups with zero.*

(2) *A z-semigroup of order* 2.

(3) $S=\{0\}\cup S'$ *where* 0 *is a zero of* S, *and* S' *is a finite simple semigroup without zero. We permit* S' *to be a one-element semigroup.*

As a special case, we get

Lemma 3. S *is a finite commutative simple semigroup without zero if and only if* S *is a finite commutative group.* S *is a finite commutative simple semigroup with zero if and only if* S *is either a z-semigroup of order* 2 *or a finite commutative group with zero adjoined.*

2. Finite simple c-indecomposable semigroups with zero. According to Rees [3], a finite simple semigroup S is completely simple, and hence it is faithfully represented as a regular matrix semigroup over a group. The defining matrix $P=(p_{\mu\lambda})$ of S is said to contain a zero if there is an element $p_{\beta\alpha}=0$ at least.

Without the condition of finiteness, we have

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1) By a proper ideal T of a semigroup S we mean a proper subset T of S such that $T\neq\{0\}$, $ST\subseteq T\neq S$, and $TS\subseteq T\neq S$.

2) Numbers in brackets [ ] refer to the references at the end of the paper.

Theorem 1. *A completely simple semigroup is c-indecomposable if its defining matrix P contains a zero.*

By Lemma 2 and Theorem 1, we obtain

Theorem 2. *A finite simple semigroup with zero is c-indecomposable if and only if it contains a zero-divisor.*

3. Finite simple *c*-indecomposable semigroups without zero. A matrix $P=(p_{\mu\lambda})$ is called to be normalized if $p_{1\lambda}=p_{\mu 1}=e$, e being a unit of G, for all λ, μ. Without loss of generality, we assume that the defining matrix is normalized. H denotes the (unique) minimal normal subgroup of G containing all non-zero elements $p_{\mu\lambda}$ of P.

Utilizing Lemma 1 and Stoll's theorem [2], we have the following

Theorem 3. *A finite simple semigroup without zero is c-indecomposable if and only if the factor group G/H is c-indecomposable.*[3]

Corollary 1. *Let S be a finite simple semigroup without zero, with a commutative ground group G. S is c-indecomposable if and only if G=H.*

Corollary 2. *A finite simple semigroup without zero is c-indecomposable if the ground group is c-indecomposable.*

Thus we have seen that the deeper study of structure of a finite simple c-indecomposable semigroup without zero is reduced to that of a finite c-indecomposable group.

4. Examples. We have obtained all the types of simple c-indecomposable semigroups of order n, $2\leqq n\leqq 10$, which we shall arrange them all in another paper. Here we show only the number of the types for each n in the following table.

Order	Number of isomorphically distinct types		Number of isomorphically and anti-isomorphically distinct types	
	Without zero	With zero	Without zero	With zero
2	2	0	1	0
3	2	0	1	0
4	3	0	2	0
5	2	2	1	2
6	4	0	2	0
7	2	8	1	4
8	5	0	3	0
9	3	16	2	9
10	4	16	2	13

3) G/H is possible to be one-element semigroup, i.e. $G=H$.

References

[1] T. Tamura: The theory of construction of finite semigroups I, Osaka Math. Jour., **8**, no. 2, 243–261 (1956).

[2] R. R. Stoll: Homomorphisms of a semigroup onto a group, Amer. Jour. Math., **73**, no. 2, 475–481 (1951).

[3] D. Rees: On semigroups, Proc. Cambridge Philos. Soc., **36**, 387–400 (1940).