1. On the Singular Integrals. V

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1. In the previous two papers [2, III, IV], we have studied the Hilbert transform from a point of view of the interpolation of operation and its applications. In [2, III] we have given a negative example as to the existence of this transformation, so we introduce a modified definition for a function of the more extensive class. In the book of N. I. Achiezer [1, p. 126] we find a modified definition, but this definition does not seem to be appropriate for the case $p>2$, because in the class L^p $(p>2)$ the Fourier transform does not necessarily exist. Here we introduce a new definition — a generalized Hilbert transform of order r:

$$\tilde{f}_r(x)=\frac{(x+i)^r}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^r}\frac{dt}{x-t}, \tag{1.1}$$

where r is any positive real number.

In particular $\tilde{f}_0(x)$ means the ordinary one. Let $f(x)$ belong to L^p $(p\geqq 1)$ and $r=n$ $(n=1,2,\cdots)$. Then we have

$$\tilde{f}_n(x)=\tilde{f}_0(x)+C_{n-1}(x+i)^{n-1}+\cdots+C_0, \tag{1.2}$$

where

$$C_n=\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^{n+1}}dt,\quad (n=0,1,2,\cdots). \tag{1.3}$$

The present paper consists of two parts. In the first part we shall treat the integrability of (1.1) after [2, III]. In the second part we shall prove the reciprocal formula, and this plays an essential role in the study of the analytic function in a half-plane, as before [2, IV].

Chapter I. Integrability of the generalized Hilbert transform

2. Let $f(x)$ be a real or complex valued measurable function over $(-\infty,\infty)$. In order to make some variety we introduce the measure function as before

$$\mu(\alpha,x)=\int_0^x\frac{dt}{1+|t|^\alpha}\quad (0\leqq\alpha<1). \tag{2.1}$$

By L^p_μ $(p\geqq 1)$ we will denote the class of functions such that

$$\left(\int_{-\infty}^{\infty}|f(x)|^p d\mu(\alpha,x)\right)^{\frac{1}{p}}=\left(\int_{-\infty}^{\infty}|f(x)|^p d\mu\right)^{\frac{1}{p}}<\infty. \tag{2.2}$$

Then if we put

$$(2.3)\qquad T_r f=\frac{\tilde{f}_r}{(x+i)^r}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^r}\frac{dt}{x-t},$$

this defines a linear operator of f, and $T_r f$ may be considered as an ordinary Hilbert transform of $f(t)/(t+i)^r$.

Thus we have immediately the following theorems.

Theorem 1. *Let* $f(t)/(t+i)^r\in L_\mu^p$ $(p>1,\ r\geqq 0,\ 0\leqq\alpha<1)$ *then the operation* $T_r f$ *can be defined and we have*

$$(2.4)\qquad \int_{-\infty}^{\infty}\frac{|\tilde{f}_r(x)|^p}{1+|x|^{rp}}d\mu\leqq A_{p,\alpha}\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+|x|^{rp}}d\mu,$$

$$(2.5)\qquad \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}_r(x)-\tilde{f}_{r,\eta}(x)|^p}{1+|x|^{rp}}d\mu=0,$$

where

$$(2.6)\qquad \tilde{f}_{r,\eta}(x)=\frac{(x+i)^r}{\pi}\int_{|x-t|>\eta}\frac{f(t)}{(t+i)^r}\frac{dt}{x-t}.$$

Theorem 2. *Let* $f(x)$ *be a function such that*

$$(2.7)\qquad \int_{-\infty}^{\infty}\frac{|f(x)|\log^+[(1+|x|^2)|f|]}{1+|x|^r}d\mu$$

where $r\geqq 0$, $0<\alpha<1$. *Then the operation* $T_r f$ *can be defined and we have*

$$(2.8)\qquad \int_{-\infty}^{\infty}\frac{|\tilde{f}_r(x)|}{1+|x|^r}d\mu\leqq A\int_{-\infty}^{\infty}\frac{|f|\log^+[(1+|x|)^{2-r}|f|]}{1+|x|^r}d\mu+B,$$

$$(2.9)\qquad \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}_{r,\eta}(x)-\tilde{f}_r(x)|}{1+|x|^r}d\mu=0,$$

where A, B, *are absolute constants.*

Theorem 3. *Let* $f(x)$ *be a function such that*

$$(2.10)\qquad \int_{-\infty}^{\infty}\frac{|f|\log^+[(1+|x|)^{2-r}|f|]}{1+|x|^r}dx<\infty$$

where $r\geqq 0$. *Then we have*

$$(2.11)\qquad \int_{-\infty}^{\infty}\frac{|\tilde{F}_r(x)|}{1+|x|^r}dx\leqq A\int_{-\infty}^{\infty}\frac{|f|\log^+[(1+|x|)^{2-r}|f|]}{1+|x|^r}dx+B,$$

$$(2.12)\qquad \lim_{\lambda,\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{F}_{r,\lambda}(x)-\tilde{F}_{r,\eta}(x)|}{1+|x|^r}dx=0,$$

where

$$(2.13)\qquad \tilde{F}_{r,\eta}(x)=\tilde{f}_{r,\eta}(x)-\frac{K_1(x)}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^r}dt,$$

$$(2.14)\qquad K_1(x)=1/x\quad \textit{if}\quad |x|\geqq 1,\ =0,\ \textit{elsewhere},$$

$$(2.15)\qquad \tilde{F}_r(x)=\lim_{\eta\to 0}\tilde{F}_{r,\eta}(x).$$

Theorem 4. *Let* $f(x)/(x+i)^r$ *belong to* L_μ $(0\leqq\alpha<1)$. *Then the* *operation* $T_r f$ *can be defined and we have*

$$\int_{-\infty}^{\infty}\frac{|\tilde{f}_r(x)|^{1-\varepsilon}}{1+|x|^{r+\delta}}d\mu \leqq \frac{A}{\varepsilon\{\delta-\varepsilon(1-\alpha)\}}\Big(\int_{-\infty}^{\infty}\frac{|f(x)|}{1+|x|^r}d\mu\Big)^{1-\varepsilon}, \tag{2.16}$$

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}_{r,\eta}(x)-\tilde{f}_r(x)|^{1-\varepsilon}}{1+|x|^{r+\delta}}d\mu=0, \tag{2.17}$$

where $0<\varepsilon<1$, $\delta>\varepsilon(1-\alpha)$ *and* A *is an absolute constant.*

3. In the sequel we also define a modified discrete transform, that is for any sequence $X=(\cdots, x_{-1}, x_0, x_1,\cdots)$ we define $\tilde{X}_r$ by the following formula:

$$\tilde{X}_r=(\cdots, \tilde{x}_{-1}^{(r)}, \tilde{x}_0^{(r)}, \tilde{x}_1^{(r)},\cdots), \tag{3.1}$$

$$\tilde{x}_n^{(r)}=(n+i)^r\sum_{m=-\infty}^{\infty}{}'\frac{x_m}{(m+i)^r}\frac{1}{n-m}, \tag{3.2}$$

where the prime means that the term $m=n$ is omitted in summation. Since $\{\tilde{x}_n^{(r)}/(n+i)^r\}$ is an ordinary discrete Hilbert operation of $\{x_n/(n+i)^r\}$, we have the following theorems:

Theorem 5. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty}\frac{|x_n|^p}{1+|n|^{rp+\alpha}}<\infty, \quad (p>1,\ r\geqq 0,\ 0\leqq\alpha<1). \tag{3.3}$$

Then $\tilde{X}_r$ *can be defined and we have*

$$\sum_{-\infty}^{\infty}\frac{|\tilde{x}_n^{(r)}|^p}{1+|n|^{rp+\alpha}}\leqq A_{p,\alpha}\sum_{-\infty}^{\infty}\frac{|x_n|^p}{1+|n|^{rp+\alpha}}. \tag{3.4}$$

Theorem 6. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty}\frac{|x_n|\log^+[(1+|n|)^{2-r}|x_n|]}{1+|n|^{r+\alpha}}<\infty \tag{3.5}$$

for $r\geqq 0$, $0<\alpha<1$. *Then the operation* $\tilde{X}_r$ *can be defined and we have*

$$\sum_{-\infty}^{\infty}\frac{|\tilde{x}_n^{(r)}|}{1+|n|^{r+\alpha}}\leqq A\sum_{-\infty}^{\infty}\frac{|x_n|\log^+[(1+|n|)^{2-r}|x_n|]}{1+|n|^{r+\alpha}}+B. \tag{3.6}$$

Theorem 7. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty}\frac{|x_n|\log^+[(1+|n|)^{2-r}|x_n|]}{1+|n|^{r}}<\infty \tag{3.7}$$

for $r\geqq 0$. *Then we have*

$$\sum_{-\infty}^{\infty}\frac{|\tilde{x}_n^{(r)*}|}{1+|n|^{r}}\leqq A\sum_{-\infty}^{\infty}\frac{|x_n|\log^+[(1+|n|)^{2-r}|x_n|]}{1+|n|^{r}}+B, \tag{3.8}$$

where

$$\tilde{x}_n^{(r)*}=\tilde{x}_n^{(r)}-\frac{1}{n}\sum_{-\infty}^{\infty}\frac{x_n}{(n+i)^r}. \tag{3.9}$$

Theorem 8. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty}\frac{|x_n|}{1+|n|^{r+\alpha}}<\infty, \quad (r\geqq 0,\ 0\leqq\alpha<1). \tag{3.10}$$

Then the operation $\tilde{X}_r$ *can be defined and we have*

$$(3.11)\qquad \sum_{-\infty}^{\infty}\frac{|\tilde{x}_n^{(r)}|^{1-\varepsilon}}{1+|n|^{r+\delta+\alpha}}\leqq\frac{A}{\varepsilon\{\delta-\varepsilon(1-\alpha)\}}\left(\sum_{-\infty}^{\infty}\frac{|x_n|}{1+|n|^{r+\alpha}}\right)^{1-\varepsilon},$$

where $0<\varepsilon<1$, $\delta>\varepsilon(1-\alpha)$ *and* A *is an absolute constant.*

Chapter II. The reciprocal formula and analytic functions in an upper half-plane

4. Let $g(x)$ be a real valued measurable function over $(-\infty, \infty)$. We introduce some notations:

$$(4.1)\qquad C_r(z, g)=\frac{(z+i)^r}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{(t+i)^r}\frac{dt}{t-z},$$

$$(4.2)\qquad P_r(z, g)=\frac{(z+i)^r}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(t+i)^r}\frac{y\,dt}{(t-x)^2+y^2},$$

$$(4.3)\qquad \tilde{P}_r(z, g)=-\frac{(z+i)^r}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(t+i)^r}\frac{(t-x)\,dt}{(t-x)^2+y^2}.$$

If we put $r=0$ we have $C(z, g)$, $P(z, g)$ and $\tilde{P}(z, g)$ in [2, IV] respectively. We have also

$$(4.4)\qquad 2C_r(z, g)=P_r(z, g)+i\tilde{P}_r(z, g).$$

By analogous arguments follows:

Theorem 9. *Let* $g(x)/(x+i)^r$ *belong to* L_μ^p $(p\geqq 1,\ 0\leqq\alpha<1)$. *Then we have*

$$(4.5)\qquad \text{(S)-}\lim_{y\to 0} P_r(z, g)=g(x)\ (y\to 0),\quad a.e.\ x,$$

$$(4.6)\qquad \lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|P_r(z, g)-g(x)|^p}{1+|x|^{rp}}d\mu=0.$$

Proof. The (4.5) is trivial. As to (4.6) we have

$$(4.7)\qquad \int_{-\infty}^{\infty}|P_r(z, g)-g(x)|^p\frac{d\mu}{1+|x|^{rp+\alpha}}$$

$$\leqq 2^p\int_{-\infty}^{\infty}\frac{|(z+i)^r-(x+i)^r|^p}{1+|x|^{rp}}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{(t+i)^r}\frac{y\,dt}{(t-x)^2+y^2}\right|^p d\mu$$

$$+2^p\int_{-\infty}^{\infty}\frac{|(x+i)^r|^p}{1+|x|^{rp}}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x+t)}{(x+t+i)^r}\frac{y\,dt}{t^2+y^2}-\frac{g(x)}{(x+i)^r}\right|^p d\mu$$

$$\leqq A_p y^p\int_{-\infty}^{\infty}\frac{|g(x)|^p}{1+|x|^{rp}}d\mu+o(1)$$

$$=o(1),\quad (y\to 0).$$

Theorem 10. *Let* $g(x)/(x+i)^r$ *belong to* L_μ^p $(p>1,\ 0\leqq\alpha<1)$, *or* $g(x)/(x+i)^r$ *and* $\tilde{g}_r(x)/(x+i)^r$ *both belong to* L_μ $(0\leqq\alpha<1)$. *Then we have also*

$$(4.8)\qquad \text{(S)-}\lim_{y\to 0}\tilde{P}_r(z, g)=\tilde{g}_r(x),\quad a.e.\ x,$$

$$(4.9)\qquad \lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{P}_r(z, g)-\tilde{g}_r(x)|^p}{1+|x|^{rp}}d\mu=0.$$

Instead of this theorem, it is enough to prove the next one:

Theorem 11. *Let* $g(x)/(x+i)^r$ *belong to* L^p_μ $(p>1,\ 0\leqq\alpha<1)$, *or* $g(x)/(x+i)^r$ *and* $\tilde{g}_r(x)/(x+i)^r$ *both belong to* L_μ $(0\leqq\alpha<1)$. *Then we have*

$$\tilde{P}_r(z,g)=P_r(z,\tilde{g}_r). \tag{4.10}$$

Proof. Since it holds that $\tilde{P}_0(z,g)=P(z,\tilde{g}_0)$, we have

$$\begin{aligned}\tilde{P}_r(z,g)&=-\frac{(z+i)^r}{\pi}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{g(u)}{(u+i)^r}\frac{du}{t-u}\Big)\frac{y\,dt}{(t-x)^2+y^2}\\&=P_r(z,\tilde{g}_r).\end{aligned} \tag{4.11}$$

By Theorems 9 and 10 we have

Theorem 12. *Under the assumptions of Theorem* 10, *we have*

$$C_r(z,g)=C_r(z,i\tilde{g}_r). \tag{4.12}$$

Theorem 13. *Under the assumptions of Theorem* 10, *we have the reciprocal formula*

$$(\tilde{\tilde{g}}_r)_r(x)=-g(x),\quad a.e.\ x. \tag{4.13}$$

We take this property as a base of our arguments as before.

5. In this section we establish the representation theorem of Cauchy and Poisson type, under giving the boundary function.

Theorem 14. *Under the assumptions of Theorem* 10,

$$f(z)=2C_r(z,g) \tag{5.1}$$

defines an analytic function on the upper half-plane and

$$\text{(S)-}\lim_{y\to 0}f(z)=f(x)=g(x)+i\tilde{g}_r(x),\quad for\quad a.e.\ x, \tag{5.2}$$

and $f(z)$ *is represented by its Cauchy and its Poisson integral respectively.*

Theorem 15. *Let* $f(z)$ *be represented by its Cauchy integral with limit function* $f(x)$, *such that* $f(x)/(x+i)^r$ *belongs to* L^p_μ $(p\geqq 1,\ 0\leqq\alpha<1)$, *then we have*

$$(\widetilde{\Re f})_r=\Im f\quad and\quad (\widetilde{\Im f})_r=-\Re f. \tag{5.3}$$

Theorem 16. *Let* $f(z)$ *be analytic in the half-plane* $y>0$. *Let* $f(z)$ *have the limit function* $f(x)$ *such that* $f(x)/(x+i)^r$ *belongs to* L^p_μ $(p>1,\ 0\leqq\alpha<1)$. *Furthermore this limit exists as an angular limit on a point of a set of* x *with a positive measure. Then* $f(z)$ *can be represented by the formula*

$$f(z)=C_r(z,f). \tag{5.4}$$

Theorem 17. *Let* $f(z)$ *be analytic in the half-plane* $y>0$ *and have the limit function* $f(x)$ *such that* $f(x)/(x+i)^r$ *belongs to* L^p_μ $(p\geqq 1,\ 0\leqq\alpha<1)$. *Then whenever* $f(z)$ *is represented by its Cauchy integral of order* r, *it is also represented by its Poisson integral of order* r *and vice versa.*

6. In this section we treat an analytic function in the upper half-plane of the so-called $\mathfrak{H}^p_\mu$ class. That is an analytic function in $y>0$ such that

$$\|f(z)\|_{p,\mu}=\left(\int_{-\infty}^{\infty}|f(x+iy)|^p\,d\mu\right)^{1/p}<\text{const.} \tag{6.1}$$

for $0<y<\infty$.

Let $f(z)$ be analytic and $f(z)/(z+i)^r$ belong to $\mathfrak{H}^p_\mu$, then if we consider $f(z)/(z+i)^{r+2}$ instead of $f(z)$ we have by similar arguments

Theorem 18. *Let $f(z)/(z+i)^r$, $r\geqq 0$ belong to $\mathfrak{H}^p_\mu$ $(p\geqq 1,\ 0\leqq\alpha<1)$ there exists a limit function $f(x)$ such that $f(x)/(x+i)^r$ belongs to L^p_μ and furthermore this limit exists as an angular limit.*

Theorem 19. *Under the assumptions of Theorem* 18, *we have*

$$f(z)=o(|z|^r),\ \text{as}\ z\to\infty,\ \text{unif. in}\ y\geqq\eta>0. \tag{6.2}$$

Theorem 20. *Under the assumptions of Theorem* 18, *we can write*

$$f(z)=B_f(z)H(z) \tag{6.3}$$

where $H(z)$ belongs to the same class of $f(z)$ and does not vanish in the upper half-plane and

$$B_f(z)=\prod_{(\nu)}\frac{z-z_\nu}{z-\bar{z}_\nu}\,\frac{\bar{z}_\nu-i}{z_\nu+i} \tag{6.4}$$

with $\{z_\nu\}$ a sequence of zeros of $f(z)$ in $y>0$. This product has properties:

$$|B_f(z)|<1 \quad \text{for all}\ y>0, \tag{6.5}$$

$$(\text{S})\text{-}\lim_{y\to 0} B_f(z)=1,\quad \text{a.e.}\ x. \tag{6.6}$$

Theorem 21. *Under the assumptions of Theorem* 18, *$f(z)$ is represented by its Cauchy and its Poisson integral. As for real part of $f(x)$ we have also*

$$f(z)=2C_r(z,\Re f)=P_r(z,\Re f)+i\widetilde{P}_r(z,\Re f). \tag{6.7}$$

Theorem 22. *Under the assumptions of Theorem* 18, *we have*

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\frac{|f(x+iy)-f(x)|^p}{1+|x|^{rp}}\,d\mu=0. \tag{6.8}$$

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