62. *Embeddings of Projective Spaces into Elliptic Projective Lie Groups*

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The real, complex and quaternion projective spaces are topologically closely connected with the classical Lie groups (orthogonal group $O(n)$, unitary group $U(n)$ and symplectic group $Sp(n)$). For example, the projective spaces can be embedded into the classical Lie groups. This inclusion map φ is defined by

$$\varphi([x_1, x_2, \cdots, x_n]) = (\delta_{ij} - 2x_i\bar{x}_j) \qquad i, j = 1, 2, \cdots, n,$$

where $|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 1$, and φ plays some important role to study the topologies of the classical Lie groups [4]. These embeddings are extendable to the case of the field of octanions (i.e. Cayley numbers). That is, in this paper, we shall show that *the octanion projective plane* Π *can be embedded into the group* F_4 which is a compact simply connected F_4-type exceptional simple Lie group.

1. Let F be the field of real numbers R, complex numbers C, quaternions Q or octernions $\mathfrak{C}$.

Let $\mathfrak{J}$ be the set of all hermitian matrices of 3 order

$$X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}$$

with coefficients in F. We define the Jacobi multiplication in $\mathfrak{J}$ by

$$X \circ Y = 1/2(XY + YX),$$

the inner product in $\mathfrak{J}$ by

$$(X, Y) = tr(X \circ Y),$$

an another multiplication in $\mathfrak{J}$ by

$$X \times Y = 2X \circ Y - tr(X)Y - tr(Y)X + (tr(X)tr(Y) - (X, Y))E^{1)}$$

and define

$$(X, Y, Z) = (X, Y \circ Z).$$

Let $A(\mathfrak{J})$ be the group of all automorphisms of $\mathfrak{J}$, i.e. $\alpha \in A(\mathfrak{J})$ is a non-singular linear transformation of $\mathfrak{J}$ which satisfies

$$\alpha(X \circ Y) = \alpha X \circ \alpha Y.$$

This group $A(\mathfrak{J})$ is characterized that the group of all non-singular linear transformations of $\mathfrak{J}$ which invariant (X, Y) and (X, Y, Z), i.e.

$$(\alpha X, \alpha Y) = (X, Y) \qquad \text{for } X, Y \in \mathfrak{J}$$
$$(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z) \qquad \text{for } X, Y, Z \in \mathfrak{J}.$$

In the case of R (resp. C, Q), for any $\alpha \in A(\mathfrak{J})$, there exists an orthogonal matrix $O \in O(3)$ (resp. unitary matrix $U \in U(3)$, symplectic

1) E is the unit matrix of 3 order.

matrix $W \in Sp(3)$) such that $\alpha X = OXO^{-1}$ (resp. $\alpha X = UXU^{-1}$ or $\alpha X = U\bar{X}U^{-1}$, $\alpha X = WXW^{-1}$) for all $X \in \mathfrak{J}$. In the case of $\mathfrak{C}$, $A(\mathfrak{J})$ is a compact connected simply connected F_4-type exceptional simple Lie group.

We refer the following lemmas [1-3].

Lemma 1. *For $X \in \mathfrak{J}$, the following 6 conditions are equivalent:*

1) *X is a non zero irreducible idempotent, i.e. $X \neq 0$, $X = X \circ X$ and if $X = X_1 + X_2$, where $X_i \in \mathfrak{J}$, $X_i \circ X_i = X_i$ $(i=1,2)$* and $X_1 \circ X_2 = 0$, then $X_1 = 0$ or $X_2 = 0$.

2) *$X = X \circ X$ and $tr(X) = 1$.*

3) *$tr(X) = tr(X \circ X) = tr((X \circ X) \circ X) = 1$.*

4) *There exists $\alpha \in A(\mathfrak{J})$ such that $X = \alpha(E_1)$.*[2]

5) *$X \times X = 0$ and $tr(X) = 1$.*

6) $\xi_1 + \xi_2 + \xi_3 = 1$, $\xi_2\xi_3 = |x_1|^2$, $\xi_3\xi_1 = |x_2|^2$, $\xi_1\xi_2 = |x_3|^2$
$\xi_3\bar{x}_3 = x_1x_2$ $\xi_1\bar{x}_1 = x_2x_3$ $\xi_2\bar{x}_2 = x_3x_1$.

Let P be the set of all elements X satisfying one of the above conditions 1)-6). P is called the projective plane over F.

Lemma 2. *$tr(\alpha X) = tr(X)$ for $\alpha \in A(\mathfrak{J})$ and $X \in \mathfrak{J}$.*

Lemma 3. *$\alpha(X \times Y) = \alpha X \times \alpha Y$ for $\alpha \in A(\mathfrak{J})$ and $X, Y \in \mathfrak{J}$.*

2. We define an inclusion map φ: $P \to A(\mathfrak{J})$ by

$$\begin{cases} \varphi(A)X = Y & \text{for } A \in P \text{ and } X \in \mathfrak{J} \\ \quad Y = 4(A \times X) \times A + 4A \circ X - 3X. \end{cases}$$

We shall show that $\varphi(A) \in A(\mathfrak{J})$, namely

$$(Y, Y) = (X, X)$$
$$(Y, Y, Y) = (X, X, X).$$

We choose $\alpha \in A(\mathfrak{J})$ such that $\alpha(A) = E_1$ by Lemma 2.4), and put $\alpha(Y) = T$, $\alpha(X) = Z$. Then it is sufficient to show

$$(T, T) = (Z, Z)$$
$$(T, T, T) = (Z, Z, Z)$$

where $T = 4(E_1 \times Z) \times E_1 + 4E_1 \circ Z - 3Z$. Computing directly

$$Z = \begin{pmatrix} \zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & \zeta_2 & z_1 \\ z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix}$$

$$E_1 \times Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_3 & -z_1 \\ 0 & -\bar{z}_1 & \zeta_2 \end{pmatrix}$$

$$(E_1 \times Z) \times E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & \zeta_3 \end{pmatrix}$$

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2) $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$E_1 \circ Z = 1/2\begin{pmatrix} 2\zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix}.$$

Therefore

$$T = \begin{pmatrix} \zeta_1 & -z_3 & -\bar{z}_2 \\ -\bar{z}_3 & \zeta_2 & z_1 \\ -z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix}.$$

Hence obviously we have $(T, T) = (Z, Z)$ and $(T, T, T) = (Z, Z, Z)$.

Next we shall show that φ is one-to-one. For $A, B \in P$, suppose that

$$\varphi(A)X = \varphi(B)X \qquad \text{for all } X \in \mathfrak{J}.$$

We may suppose that $B = E_1$, namely

$$4(A \times X) \times A + 4A \circ X - 3X = 4(E_1 \times X) \times E_1 + 4E_1 \circ X - 3X.$$

Put $X = E_1$, then

$$4(A \times E_1) \times A + 4A \circ E_1 - 3E_1 = E_1.$$

Compare the (1, 1)-element, then we have

$$4\alpha_2^2 + 4\alpha_3^2 + 8|a_1|^2 + 4\alpha_1 - 3 = 1$$
$$\alpha_2^2 + \alpha_3^2 + 2\alpha_2\alpha_3 + \alpha_1 = 1$$
$$(\alpha_2 + \alpha_3)^2 = 1 - \alpha_1$$
$$(1 - \alpha_1)^2 = 1 - \alpha_1.$$

Hence we have $\alpha_1 = 1$ or $\alpha_1 = 0$.

Next put $X = E_2$, then

$$4(A \times E_2) \times A + 4A \circ E_2 - 3E_2 = E_2.$$

Compare the (2, 2)-element, then we have $\alpha_2 = 1$ or 0. Analogously we have $\alpha_3 = 1$ or 0. Hence $A = E_1$, E_2 or E_3. If $A = E_2$, then put $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then (1, 3)-element of $\varphi(E_2)X = 1$ and that of $\varphi(E_1)X = -1$. This is a contradiction. Therefore we have $A = E_1$.

Theorem. *The projective plane P can be embedded into the group $A(\mathfrak{J})$. Especially, the octanion projective plane Π can be embedded into the exceptional simple Lie group F_4.*

References

[1] Freudenthal, H.,: Oktaven, Ausnahmengruppen und Oktavengeometrie, Math. Inst., Rijks Univ. te Utrecht (1951).
[2] ——: Zur ebene Oktavengeometrie, Indagations Math., **15**, 195–200 (1953).
[3] Yokota, I.,: On the cell structure of the octanion projective plane Π, Jour. Inst. Poly., Osaka City Univ., **6**, 31–37 (1955).
[4] ——: On the homology of classical Lie groups, ibid., **8**, 93–120 (1957).