82. A Remark on the Abstract Analyticity in Time for Solutions of a Parabolic Equation

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(Comm. by K. KUNUGI, M.J.A., July 13, 1959)

1. Consider an equation of evolution

(1.1) $$du/dt = A(t)u$$

where the differential operator

$$A(t)=\sum_{i,j=1}^{m} a^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{m} b^i(t,x)\frac{\partial}{\partial x_i}+c(t,x)$$

is elliptic on a domain G of an m-dimensional Euclidean space. As for the case when all the coefficients of $A(t)$ are independent of t, K. Yosida [7] extended the result of S. Itô and H. Yamabe [5, 6].

In the present note the author will give a direct proof of K. Yosida's result under an assumption that all the coefficients $a^{ij}(t,x)$, $b^i(t,x)$, $c(t,x)$ are uniformly analytic in t for any x in G.

The method is based upon the idea of C. B. Morrey and L. Nierenberg [2]. The result, which is applicable to the unique continuation problem of (1.1) [1], is obviously extended with respect to certain distribution solutions of generalized parabolic equations [4].

2. For the sake of simplicity, we shall discuss the case $G=E^m$ and assume that the real coefficients $a^{ij}(t,x)$, $b^i(t,x)$, and $c(t,x)$ are sufficiently differentiable such that

$$D_x^{(k)}D_t^{(p)}a^{ij}(t,x),\quad D_x^{(k')}D_t^{(p)}b^i(t,x),\quad D_t^{(p)}c(t,x)$$
$$(k=0,1,2;\ k'=0,1;\ p=0,1,2,3,\cdots)$$

are continuous over $[-1,1]\times E^m$, and that there are two positive numbers L and K such that

(2.1) $$\underset{\substack{k=0,1\\ i,j=1,2,\cdots,m}}{\mathrm{Max}}\ \{|D_x^{(k)}a^{ij}(t,x)|,\ |b^i(t,x)|,\ |c(t,x)|\}\leqq L,$$

(2.2) $$\underset{\substack{p=0,1,2,\cdots\\ i,j=1,2,\cdots,m}}{\mathrm{Max}}\ \{|D_t^{(p)}a^{ij}(t,x)|,\ |D_t^{(p)}b^i(t,x)|,\ |D_t^{(p)}C(t,x)|\}\leqq Lp!\,K^p$$

for any $x\in E^m$, $t\in(-1,1)$. Moreover there are two positive γ and δ such that

(2.3) $$\gamma\sum_{i=1}^m \xi_i^2 \geqq \sum_{i,j=1}^m a^{ij}(t,x)\xi_i\xi_j \geqq \delta\sum_{i=1}^m \xi_i^2$$

for any $x\in E^m$, $t\in(-1,1)$ and for any real $\xi=(\xi_1,\cdots,\xi_m)$. Set

$$\|f(t,x)\|_r^2=\int_{-r}^{r}\int_{E^m}|f(t,x)|^2\,dxdt$$

and

$$\overline{A(t)}=A(t)-\alpha$$

for sufficiently large but fixed α. Furthermore for $R<1$ define e_p, d_p, $M_{R,p}$, and $N_{R,p}$ by

$$e_p(f,r)^2=\|D_t^{(p)}f\|_r^2,\ d_p(u,r)^2=\|D_t^{(p+1)}u\|_r^2+\|\overline{A(0)}D_t^{(p)}u\|_r^2$$

$$M_{R,p}(f)=(p!)^{-1}\sup_{R/2<r<R}(R-r)^p e_p(f,r)$$

$$N_{R,p}(u)=(p!)^{-1}\sup_{R/2<r<R}(R-r)^p d_p(u,r).$$

Then according to Morrey and Nierenberg, there exist two positive K_1 and K_2 for sufficiently smooth functions u, f satisfying $(d/dt-\overline{A(t)})u=f$ $(t\in(-1,1))$ such that for any integer $p>0$

$$(2.4)_p \qquad N_{R,p}(u)\leqq K_2\{M_{R,p}(f)+LK_1RN_{R,p}(u)+L\sum_{t=1}^{p}(K_1R)^tN_{R,p-t}(u)+\sum_{t=0}^{p-1}N_{R,t}(u)\}$$

where we assume that K_1 is sufficiently large $(K_1>K)$, $K_2LK_1R<1/2$ and $R<R_1=\min(1/K_2LK_1, 1)$.

The inequality $(2.4)_p$ follows from the inequality

$$\|(d/dt-\overline{A(0)})u\|^2\geqq K_2'(\|du/dt\|^2+\|\overline{A(0)}u\|^2)$$

where u is a sufficiently smooth function such that the carrier of u is contained in $(-1,1)\times E^m$ and where K_2 and K_2' depend only upon α,γ,δ, L and K_1 (for the case of a bounded domain, see [2] and [3]).

From (2.4) we see the following

Theorem. The solution $u(t)$ of (1.1) is an analytic function of a neighbourhood of 0 into $L_2(E^m)$ wherever $u(t)$ is a continuously differentiable function of $(-1,1)$ into $L_2(E^m)$.

For by using convolution operator and the difference operator we see from $(2.4)_1$ that the continuously differential solution $u(t)$ of (1.1) is infinitely differentiable in t such that both $\|D_t^{(p)}u\|_{1/2}$ and $\|\overline{A(0)}D_t^{(p)}u\|_{1/2}$ are finite [4].

Accordingly, again using $(2.4)_p$, there are two positive λ and M such that

$$\|D_t^{(p)}u\|_R=Mp!\,\lambda^p$$

for any $p>0$ and for all sufficiently small R, which implies the analyticity of $u(t)$ in time at $t=0$.

Corollary. If a continuously differentiable solution $u(t)$ of (1.1) for $t>0$ satisfies the condition: for a fixed $t_0>0$ $u(t_0,x)=0$ on an open set $G\subset E^m$, then $u(t,x)=0$ for any $t>0$ and any $x\in G$.

Because $f(t)=\int u(t,x)\varphi(x)dx$, for a $\varphi(x)$ in $\mathfrak{D}(G)$ is analytic in $t>0$ by the theorem above-mentioned and $D_t^{(p)}f(t_0)=0$ for any $p\geqq 0$ [4]. Therefore $f(t)\equiv 0$ $(t>0)$ and hence $u(t,x)\equiv 0$ for $(t>0,\ x\in G)$.

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