131. On Some Properties of Intermediate Logics

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In [1] I investigated inclusion and non-inclusion between certain intermediate predicate logics. The purpose of this note is to prove some properties of intermediate logics. We use in this note the same notations as in [1] without definitions.

1. Interpretation of classical logic. THEOREM 1. *LK° and LMK° are minimal in the set of all predicate logics which have the properties* (I) *and* (II) *respectively.*

(I) *For any K-provable sequent $\Gamma\to E$, the sequent $\neg\neg\Gamma\to\neg\neg E$ is provable.*

(II) *For any K-provable sequent $\Gamma\to\Delta$, the sequent $\neg\neg\Gamma\to\neg\neg\Delta$ is provable.*

PROOF. First we prove that (I) and (II) hold in LK° and LMK° respectively. For (I) we use as a deductive system of K-provable sequents the rules of inference in Gentzen's LJ [2] with the axiom schemes $A\to A$ and $\to A\vee\neg A$. As for $A\to A$ and $\to A\vee\neg A$ (I) clearly holds. Let us assume that (I) holds for the upper sequent(s) of any rule of inference. This is proved by an inductive method. As an example, we treat $\to\forall$. From $\neg\neg\Gamma\to\neg\neg A(a)$ we obtain $\neg\neg\Gamma\to\forall x\neg\neg A(x)$. Since $\forall x\neg\neg A(x)\to\neg\neg\forall xA(x)$ is provable in LK°, we obtain $\neg\neg\Gamma\to\neg\neg\forall xA(x)$, which shows that (I) holds for the lower sequent of $\to\forall$.

For (II) we use Gentzen's LK as a deductive system of K-provable $\Gamma\to\Delta$. Only $\to\neg$ and $\to\forall$ are the rules of inference which use MK° in a proof of LMK°. We prove (II) only for $\to\forall$. From $\neg\neg\Gamma\to\neg\neg\Delta$, $\neg\neg A(a)$ we obtain $\neg A(a)$, $\neg\neg\Gamma\to\neg\neg\Delta$ and hence $\exists x\neg A(x)$, $\neg\neg\Gamma\to\neg\neg\Delta$. Thence $\neg\neg\exists x\neg A(x)$, $\neg\neg\Gamma\to\neg\neg\Delta$ is provable. Hence, by applying a cut with this sequent and MK° as the upper sequents of the cut, we obtain $\neg\neg\Gamma\to\neg\neg\Delta$, $\neg\neg\forall xA(x)$.

Secondly, let us assume that LZ and LY have the properties (I) and (II) respectively. Since $\to\forall x(A(x)\vee\neg A(x))$ is K-provable, $\to\neg\neg\forall x(A(x)\vee\neg A(x))$ is Z-provable and hence $LZ\supseteq LK^\circ$. In the same way we see that $\to\neg\neg\forall xA(x)$, $\neg\neg\exists x\neg A(x)$ is Y-provable and hence $LY\supseteq LMK^\circ$ since $\to\forall xA(x)$, $\exists x\neg A(x)$ is K-provable.

2. Decomposition of sequent scheme. We mean by $A\leftrightarrow B$ that both $A\to B$ and $B\to A$. In LJ' the following equivalences are provable.

i) $\neg(A\vee B)\leftrightarrow\neg A\wedge\neg B$, $\neg(A\supset B)\leftrightarrow\neg\neg A\wedge\neg B$,
$\neg(A\wedge B)\leftrightarrow A\supset\neg B$, $\neg\neg\neg A\leftrightarrow\neg A$, $A\supset(B\supset C)\leftrightarrow(A\wedge B)\supset C$,
$(A\vee B)\supset C\leftrightarrow(A\supset C)\wedge(B\supset C)$, $A\supset(B\wedge C)\leftrightarrow(A\supset B)\wedge(A\supset C)$.

ii) $\neg\forall xA(x)\leftrightarrow\forall xA(x)\supset\neg\exists xA(x)$, $\neg\exists xA(x)\leftrightarrow\forall x\neg A(x)$,
$A\wedge\forall xB(x)\leftrightarrow\forall x(A\wedge B(x))$, $A\wedge\exists xB(x)\leftrightarrow\exists x(A\wedge B(x))$,
$A\vee\exists xB(x)\leftrightarrow\exists x(A\vee B(x))$, $A\supset\forall xB(x)\leftrightarrow\forall x(A\supset B(x))$,
$\exists xA(x)\supset B\leftrightarrow\forall x(A(x)\supset B)$, $\forall xA(x)\wedge\forall xB(x)\leftrightarrow\forall x(A(x)\wedge B(x))$,
$\exists xA(x)\vee\exists xB(x)\leftrightarrow\exists x(A(x)\vee B(x))$.

iii) $A\wedge B, \Gamma\to\Delta$ is equivalent to $A, B, \Gamma\to\Delta$.
$A\vee B, \Gamma\to\Delta$ is equivalent to $A, \Gamma\to\Delta$ and $B, \Gamma\to\Delta$.
$\Gamma\to\Delta, A\wedge B$ is equivalent to $\Gamma\to\Delta, A$ and $\Gamma\to\Delta, B$.
$\Gamma\to\Delta, A\vee B$ is equivalent to $\Gamma\to\Delta, A, B$.

We say that a sequent scheme Y is *decomposed* in LX into a finite number of sequent schemes $Z_1,\cdots,Z_n$, if and only if for any i $(1\leqq i\leqq n)$ Z_i is (X, Y)-provable and Y is $(X, Z_1,\cdots,Z_n)$-provable.

THEOREM 2. *Any sequent scheme can be decomposed in LJ' into a finite number of sequent schemes, in which $\wedge$ and $\vee$ do not occur as outermost logical symbols of sequent formulas and $\neg$ occurs only as an innermost logical symbol.*

The following equivalences are provable respectively;

iv) in LM $A\supset\neg B\leftrightarrow\neg A\vee\neg B$, $\neg A\supset B\leftrightarrow\neg\neg A\vee B$,
$A\supset(B\vee\neg C)\leftrightarrow(A\supset B)\vee\neg C$, $(\neg A\wedge B)\supset C\leftrightarrow\neg\neg A\vee(B\supset C)$;

v) in LP_2 $(A\wedge B)\supset C\leftrightarrow(A\supset C)\vee(B\supset C)$,
$A\supset(B\vee C)\leftrightarrow(A\supset B)\vee(A\supset C)$,
$A\vee B\leftrightarrow((A\supset B)\supset B)\vee((B\supset A)\supset A)$;

vi) in LEK° $\neg\forall xA(x)\leftrightarrow\exists x\neg A(x)$;

vii) in LD $\forall xA(x)\vee B\leftrightarrow\forall x(A(x)\vee B)$;

viii) in LF $\forall xA(x)\supset B\leftrightarrow\exists x(A(x)\supset B)$;

ix) in LG $B\supset\exists xA(x)\leftrightarrow\exists x(B\supset A(x))$.

These logics are subsystems of LFG.

THEOREM 3. *Any sequent scheme can be decomposed in LFG into a finite number of sequent schemes in which the antecedent is empty and the succedent consists of only one formula of Skolem normal form in which the logical symbol $\vee$ does not occur.*

PROOF. By virtue of the above equivalences, any sequent scheme can be decomposed into a finite number of sequent schemes of form $F\to E$ where the logical symbol $\vee$ occurs neither in F nor in E. Then the sequent schemes obtained by this decomposition can be transformed, using the equivalence of $F\to E$ to $\to F\supset E$, to the form $\to S$ where S is a formula in which $\vee$ does not occur. In terms of ii) and vi)–ix) S can be transformed to the prenex form S'. Since $\to A(a)$ is equivalent to $\to\forall xA(x)$ where a does not occur in $A(x)$, we assume that no free individual variable occurs in S'.

Let S' be the form $\exists x_1 \cdots \exists x_n \forall y G(x_1, \cdots, x_n, y)$, $n \geq 0$, where $G(x_1, \cdots, x_n, y)$ is a prenex form containing only the distinct free individual variables $x_1, \cdots, x_n, y$. Then we replace S' by $\exists x_1 \cdots \exists x_n \; \exists y \forall z((G(x_1, \cdots, x_n, y) \supset H(x_1, \cdots, x_n, y)) \supset H(x_1, \cdots, x_n, z))$, which is denoted by S'', where H is a predicate variable with exactly $n+1$ argument-places which does not occur in G. Then $\rightarrow S'$ is equivalent to $\rightarrow S''$ in LFG. Next we transform S'' to the prenex form and repeat this procedure. Then we obtain the Skolem normal form of S. Hence the theorem has been proved.

References

[1] T. Umezawa: On logics intermediate between intuitionistic and classical predicate logic, J. Symbolic Logic, **24** (1959) (to appear).

[2] G. Gentzen: Untersuchungen über das logische Schliessen, Math. Zeitschr., **39**, 176–210; 405–431 (1934–1935).