19. Triviality of the mod p Hopf Invariants

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In this note we shall extend Adams' result[1] to the mod p case.

1. Let p be an odd prime; let A be the Steenrod algebra over Z_p. An A-module is to be a graded left module over the graded algebra A. For each integer $k\geqq 0$, define C_k to be the free A-module generated by symbols $[\mathcal{P}^{p^i}]$ of degree $2p^i(p-1)$ $(i=0,1,\cdots,k)$ and $[\Delta]$ of degree one. C_k may be considered as a submodule of C_l for $k<l$, and the inductive limit $\bigcup_k C_k$ is denoted by C. Define $d: C\to A$ to be the A-map of degree zero such that $d[\Delta]=\Delta$ and $d[\mathcal{P}^{p^i}]=\mathcal{P}^{p^i}$ $(i=0,1,\cdots)$, where $\mathcal{P}^{p^i}$ denotes the reduced power and Δ denotes the Bockstein operator.

2. We call a homogeneous element of Ker d a *d-cycle*. A d-cycle Z may be written in such a way as $\alpha_k[\mathcal{P}^{p^k}]+\alpha_{k-1}[\mathcal{P}^{p^{k-1}}]+\cdots+\alpha_0[\mathcal{P}^1]+\alpha_\Delta[\Delta]$, of which $\alpha_k[\mathcal{P}^{p^k}]$ $(\alpha_k\neq 0)$ is called the leading term of Z.

We choose specific d-cycles (occasionally indicated only by their leading terms) as follows:

$$U_0=\Delta[\Delta],\ V_0=(2\mathcal{P}^1\Delta-\Delta\mathcal{P}^1)[\mathcal{P}^1]-2\mathcal{P}^2[\Delta],\ W_0=\mathcal{P}^{p-1}[\mathcal{P}^1],$$

$$Z_k=\Delta[\mathcal{P}^{p^k}]+\cdots \qquad (k\geqq 1),$$

$$Z_{i,k}=\mathcal{P}^{p^i}[\mathcal{P}^{p^k}]+\cdots \qquad (0\leqq i\leqq k-2),$$

$$U_k=\mathcal{P}^{2p^{k-1}}[\mathcal{P}^{p^k}]+\cdots \qquad (k\geqq 1),$$

$$V_k=c(2\mathcal{P}^{p^k+p^{k-1}}-\mathcal{P}^{p^k}\mathcal{P}^{p^{k-1}})[\mathcal{P}^{p^k}]+\cdots \qquad (k\geqq 1),$$

$$W_k=c(\mathcal{P}^{p^k(p-1)})[\mathcal{P}^{p^k}]+\cdots \qquad (k\geqq 1),$$

where c is the conjugation.[2] We call these *basic* d-cycles.

Lemma. *$C_k\frown$Ker d is generated by the basic d-cycles as an A-module.*

This lemma follows from Proposition 1.7 of Toda's paper.[3]

To each basic d-cycle Z corresponds a *basic* (stable secondary cohomology) operation Φ_Z. Among the basic secondary operations, only the followings are of degree even:

Φ_{V_0}, of degree $4(p-1)$, and Φ_{Z_k}, of degree $2p^k(p-1)$ $(k\geqq 1)$.

3. We shall state a proposition which is a generalization of

1) J. F. Adams: On the non existence of elements of Hopf invariant one, Bull. Amer. Math. Soc., **64**, 279-282 (1958).

2) J. Milnor: The Steenrod algebra and its dual, Ann. of Math., **67**, 150-171 (1958).

3) H. Toda: p-primary components of homotopy groups, I. Exact sequences in Steenrod algebra, Memoirs of the College of Sci., Univ. of Kyoto, ser. A, **31**, Math., no. 2, 129-142 (1958); II. mod p Hopf invariant, ibid., **31**, 143-160 (1958).

Theorem 6.3 of Peterson-Stein.[4)]

Proposition. *Let $\sum_{i=1}^{n} \alpha_i \beta_i = 0$ be a relation in the Steenrod algebra A, where α_i, β_i are of degree positive. Let $f : X \to Y$ be a map. Let $u \in H^N(Y, Z_p)$ be a cohomology class such that $\alpha_i \beta_i u = 0$ $(i=1,\cdots,n)$ and $f^* \beta_i u = 0$ $(i=1,\cdots,n)$. Then there exists a stable secondary operation Φ corresponding to the relation $\sum \alpha_i \beta_i = 0$, and we have*

$$\Phi f^* u = -\sum_{i=1}^{n} \alpha_{i,f}(\beta_i u) \qquad \mathrm{mod}\ \mathrm{Im}\, f^* + \sum \mathrm{Im}\, \alpha_i,$$

where $\alpha_{i,f}$ denotes the functional operation.[5)]

4. We shall calculate the basic secondary operations Φ_{V_0} and Φ_{Z_k} $(k \geqq 1)$ in the infinite dimensional complex projective space P.

Let $y \in H^2(P, Z_p)$ be a generator of the cohomology ring of the space P. $\Phi_{V_0}(y^r)$ is defined only if $r \equiv 0 \mod p$ and $\Phi_{Z_k}(y^r)$ is defined only if $r \equiv 0 \mod p^{k+1}$.

Theorem 1.

$$\Phi_{V_0}(y^{pn}) = n y^{pn+2(p-1)} \qquad \text{mod } zero,$$

$$\Phi_{Z_k}(y^{p^{k+1}n}) = -n y^{p^{k+1}n + p^k(p-1)} \qquad \text{mod } zero \ (k \geqq 1).$$

In the proof of this theorem we make essential use of the proposition in the preceding section. There is another method according to Adams[6)] making use of a formula for the composite operation $\Phi_{Z_k} \mathcal{P}^{p^k(p-1)}$, but it is rather complicated for calculation.[*)]

5. We shall state the conclusion which follows from Adams' Theorem 3[7)] and the above Theorem 1.

Theorem 2. *For each $k \geqq 0$, the following formula* k) *holds for classes u such that $\Delta u = 0$, $\mathcal{P}^{p^i} u = 0$ $(i=0,\cdots,k)$ and modulo a certain subgroup Q_k:*

0) $\mathcal{P}^p u = \Delta \Phi_{W_0} u + \mathcal{P}^{p-2} \Phi_{V_0} u,$

1) $\mathcal{P}^{p^2} u = \Delta \Phi_{W_1} u + c(\mathcal{P}^{p(p-2)+1}) \Delta \mathcal{P}^{p-3} \Phi_{V_1} u - c(\mathcal{P}^{p(p-1)}) \Phi_{Z_1} u + \sum \alpha_* \Phi_{Z_*} u,$

k) $\mathcal{P}^{p^{k+1}} u = \Delta \Phi_{W_k} u - c(\mathcal{P}^{p^k(p-1)-p+1}) \Delta \mathcal{P}^{p-2} \Phi_{Z_{0,k}} u - c(\mathcal{P}^{p^k(p-1)}) \Phi_{Z_k} u + \sum \alpha_* \Phi_{Z_*} u \qquad (k \geqq 2),$

where $\alpha_* \in A$ and Z_* run over basic d-cycles belonging to C_{k-1}.

As a consequence we have

Theorem 3. *The* mod p *Hopf invariant*[8)]

$$H_p^{(k)} : \pi_{N+2p^k(p-1)-1}(S) \to Z_p$$

is trivial for each $k \geqq 1$.

4) F. P. Peterson and N. Stein: Secondary cohomology operations: two formulas, Amer. J. M., **81**, 281-305 (1959).

5) N. E. Steenrod: Cohomology invariants of mappings, Ann. of Math., **50**, 954-988 (1949).

6) Adams 1).

7) Adams 1).

8) Toda 3), II.

*) *Added in proof.* By such a method, Prof. T. Yamanoshita has also obtained the same result as ours.