18. A Continuity Theorem in the Potential Theory

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Introduction. Let Ω be a locally compact separable metric space and let Φ be a positive symmetric kernel satisfying the continuity principle, that is, let Φ be a real-valued continuous function defined on the product space $\Omega\times\Omega$ such that

1° $0<\Phi(P,Q)\leq+\infty$,

2° $\Phi(P,Q)$ is finite except at most at the points of diagonal set of $\Omega\times\Omega$,

3° $\Phi(P,Q)=\Phi(Q,P)$,

4° for any compact set $K\subset\Omega$ and for any positive number ε, there is a compact set L such that

$$\Phi(P,Q)<\varepsilon \quad \text{on } K\times(\Omega-L),$$

5° if a potential U^μ of a positive measure μ with compact support S_μ is finite and continuous as a function on the support S_μ, then it is continuous in Ω, where the potential U^μ is defined by

$$U^\mu(P)=\int\Phi(P,Q)d\mu(Q).$$

It is known that every potential U^μ of a positive measure with compact support is quasi-continuous in Ω, that is, for any positive number ε, there is an open set G_ε such that $\text{cap}(G_\varepsilon)\leq\varepsilon$ and U^μ is finite and continuous as a function on $\Omega-G_\varepsilon$. This is called an "in the large" continuity theorem. In this note we communicate an "in the small" continuity

Theorem. *Let μ be a positive measure with compact support. Then at any point P except at most at the points of a polar set, there exists an open set $G(P)$, thin at P, such that the restriction of U^μ to $\Omega-G(P)$ is finite and continuous at P.*

This theorem was proved by Deny [3] in the case of the Newtonian potentials in the m-dimensional Euclidean space. Recently Smith [6] has remarked that this is valid for the potentials of order α, $0<\alpha<m$.

1. **Capacities.** A set $E\subset\Omega$ is called a *polar set* if it is contained in some $I_\mu=\{P: U^\mu(P)=+\infty\}$, where μ is a positive measure of total measure finite. We denote by $\mathfrak{P}$ the family of all polar sets. For any set X we put

$$\mathfrak{F}_X=\{\mu\geq0;\ \mu(\Omega)<+\infty,\ U^\mu\geq1 \text{ on } X \text{ except } E\in\mathfrak{P}\},$$

$$f(X)=\begin{cases}\inf_{\mu\in\mathfrak{F}_X}\mu(\Omega) & \\ +\infty & \text{if } \mathfrak{F}_X \text{ is empty,}\end{cases}$$

$\mathrm{cap}_i(X) = \sup f(K)$ where K ranges over the class of all compact sets contained in X,

$\mathrm{cap}_e(X) = \inf \mathrm{cap}_i(G)$ where G ranges over the class of all open sets containing X.

The set functions $\mathrm{cap}_i(X)$ and $\mathrm{cap}_e(X)$ are called the inner and outer capacities of X, respectively. A set A is said to be capacitable when its inner capacity coincides with its outer capacity, and we denote by $\mathrm{cap}(A)$ the common value. Evidently every open set is capacitable.

We have obtained the following propositions in the preceding paper [4].

Proposition 1. *For any set X $f(X) = \mathrm{cap}_e(X)$.*

Proposition 2. *A set E is of outer capacity zero if and only if it is a polar set.*

Proposition 3. *Suppose that a sequence $\{\mu_n\}$ of positive measures converges vaguely to μ and that the total measures $\mu_n(\Omega)$ are bounded. Then*

$$U^{\mu} = \underline{\lim}\, U^{\mu_n}$$

in Ω with a possible exception of a polar set.

2. The fine topology. Let P_0 be an arbitrary point in Ω. A subset N of Ω is called a *fine* neighborhood of P_0 if it contains some

$$\omega(P_0) \frown \{P;\, U^{\mu}(P) < U^{\mu}(P_0) + \rho\},$$

where $\omega(P_0)$ is a neighborhood of P_0 with respect to the original topology in Ω, $\mu \geq 0$, $U^{\mu}(P_0) < +\infty$ and $\rho > 0$. We shall denote by $\mathfrak{N}(P_0)$ the family of all fine neighborhoods of P_0. It is easily seen that $\mathfrak{N}(P_0)$ has the following properties:

i) If $N_1 \in \mathfrak{N}(P_0)$ and $N_2 \supset N_1$, then $N_2 \in \mathfrak{N}(P_0)$.

ii) Every $N_1 \in \mathfrak{N}(P_0)$ contains P_0.

iii) If N_1 and N_2 belong to $\mathfrak{N}(P_0)$, then $N_1 \frown N_2$ belongs to $\mathfrak{N}(P_0)$.

iv) If $N_1 \in \mathfrak{N}(P_0)$, then there is an $N_2 \in \mathfrak{N}(P_0)$ such that $N_1 \in \mathfrak{N}(P)$ for every $P \in N_2$.

On account of these properties a topology is defined in Ω by means of $\mathfrak{N}(P)$, $P \in \Omega$. This topology is called the fine topology and denoted by $\mathfrak{T}_f$. Evidently the fine topology is stronger than the original one.

Theorem 1. *Every potential U^{μ} of a positive measure is continuous with respect to the fine topology at any point P where $U^{\mu}(P)$ is finite. The fine topology is the weakest among those topologies $\mathfrak{T}$, stronger than the original one in Ω, with respect to which every potential is continuous at any point where it is finite.*

Remark. If $\mathfrak{T}$ is not stronger than the original one, then it is not necessarily stronger than $\mathfrak{T}_f$.

3. Thin sets. **Definition.** A set A is called *thin* at P_0 when P_0 is an isolated point of $A \smile \{P_0\}$ with respect to the fine topology.

As to thin sets we refer to Brelot [1], Choquet [2] and Ohtsuka

[5] as well as papers cited in the introduction.

Immediately we have:

1) If A is thin at P_0 and $A \supset B$, then B is thin at P_0, and 2) if A and B are thin at P_0, then $A \cup B$ is thin at P_0.

The following is an answer to a part of the question raised by Choquet [2].

Theorem 2. *If E is a polar set, then E is thin at every point of Ω.*

Corollary. *Let E be a polar set and let A be thin at P_0. Then $A \cup E$ is thin at P_0.*

4. Continuity "in the small". By Propositions 1 and 3 and the corollary of Theorem 2 we can prove

Theorem 3. *Let G_n $(n=1, 2, \cdots)$ be an open set such that* $\mathrm{cap}(G_n) \to 0$. *Then there exists a polar set E such that at any point $P \in \Omega - E$ some G_n is thin.*

Our "in the small" continuity theorem follows from Theorem 3.

References

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