16. A Note on Algebras of Unbounded Representation Type

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Let A be a finite dimensional associative algebra over a field P. Recently J. P. Jans [2] has generalized R. M. Thrall's theorem concerned with algebras of unbounded representation type under a condition which can be applied to the case where P is an algebraically closed field. Jans's theorem is, however, a generalization of the condition given by T. Nakayama [4], which is also a generalization of the result of H. Brummund [1] (cf. also [3]). In this paper by a slight improvement of the proofs of Brummund the author will show that the theorem of Jans holds without the condition which is quoted above.

Except the following lemma which may be said to be our main device, the proofs in this paper are similar to those of Brummund, but for the sake of completeness and of reader's convenience we shall repeat them.

Lemma 1. *Let Q_κ and Q_λ be quasi-fields and M a left Q_κ-, right Q_λ-module*[1] *such that $(xu)y=x(uy)$ for $x\in Q_\kappa$, $y\in Q_\lambda$ and $u\in M$. If the left dimension of M over Q_κ as well as the right dimension of M over Q_λ is not less than two, then we can select two elements u and v such that they are independent (to each other) over Q_κ and Q_λ respectively.*

Proof. From the assumption we have

$$\begin{aligned} M&=Q_\kappa u+Q_\kappa u_1+\cdots \\ &=uQ_\lambda+u_2Q_\lambda+\cdots\ . \end{aligned}$$

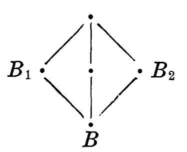
Then it is sufficient to prove the lemma for the case where $u_1\in uQ_\lambda$ and $u_2\in Q_\kappa u$. Put now $v=u_1+u_2$; then u and v satisfy our request. For if $v\in uQ_\lambda$, then $u_2\in uQ_\lambda$ and this is a contradiction. Similarly we can prove that $v\notin Q_\kappa u$ and the proof is completed.

Theorem 2. *If A has an infinite*[2] *two sided ideal lattice then A is of unbounded type.*

Proof. Let A have an infinite two sided ideal lattice; then the lattice is not distributive and contains a projective root (cf. [6], at the following diagram (projective root) B_1, B_2 and B are two sided ideals of A, and B_1 and B_2 cover B). Since the representations of the

1) All modules considered in this paper are unitary.

2) For the case of P being algebraically closed, in Theorem 2 "infinite two sided ideal lattice" can be replaced by "infinite one sided ideal lattice", but generally this replacement is not possible, for an algebra of strong left cyclic representation type may have an infinite left ideal lattice but is not of unbounded type (cf. [5, 7, 8]).



quotient ring A/B are considered as the representations of A, without loss of generality we may assume $B=(0)$. Then $B_1+B_2=B_1\oplus B_2\subset l(N)\frown r(N)$,[3] where N is the radical of A, and $B_1\approx B_2$. If we put $l(N)\frown r(N)=M$ and $A/N=\bar{A}$, there exist primitive idempotents e_κ, e_λ of A such that

$$(e_\kappa Me_\lambda:\bar{e}_\kappa\bar{A}\bar{e}_\kappa)_l\geqq 2 \quad\text{and}\quad (e_\kappa Me_\lambda:\bar{e}_\lambda\bar{A}\bar{e}_\lambda)_r\geqq 2.$$

Then we can choose elements u and v of $e_\kappa Me_\lambda$ as in Lemma 1. According to a direct decomposition such that $Me_\lambda=Ae_\kappa ue_\lambda\oplus Ae_\kappa ve_\lambda\oplus\cdots\oplus M'e_\lambda$, $e_\kappa M'e_\lambda=0$, let us denote by $Ae_\lambda h$ the quotient left A-module $Ae_\lambda/M'e_\lambda$ and by $Ae_\lambda h\varepsilon_\nu$, $\nu=1,2,\cdots,n$, n-copies of $Ae_\lambda h$. Since $Auh\approx Avh\approx\bar{A}\bar{e}_\kappa$, by identifying $xue_\lambda h\varepsilon_\nu=xve_\lambda h\varepsilon_{\nu+1}$, for all $x\in A$, $\nu=1,2,\cdots,n-1$, we obtain an interlacing module $H^{(n)}$. In the following we shall prove for an arbitrary integer n that $H^{(n)}$ is indecomposable. To this aim we shall first prove, similarly as Brummund,

Lemma 3. *If the composition length of $Ae_\lambda h$ is equal to m and if L is a submodule of $H^{(n)}$ such that the composition length of $L+NH^{(n)}/NH^{(n)}$ is t, then the composition length of L is at least $tm-(t-1)$.*

Proof of Lemma 3. Case (1), $t=1$. In this case without losing of generality we may assume that a generator α of L is expressed as follows:

$$\alpha=e_\lambda h\varepsilon_i\alpha_i+\sum_{j\neq i}x_je_\lambda h\varepsilon_j\alpha_j,$$

where α_i,α_j are injections of $Ae_\lambda h\varepsilon_i$, $Ae_\lambda h\varepsilon_j$ into $H^{(n)}$, and $x_j\in e_\lambda Ae_\lambda$, especially x_j for $j<i$ belong to $e_\lambda Ne_\lambda$. If X is an element of the kernel of the natural homomorphism $Ae_\lambda\to Ae_\lambda\alpha$, from the definition of interlacing we get

$$X\equiv(xue_\lambda+yve_\lambda)\ \mathrm{mod.}\ M'$$

and
$$X\alpha=(xue_\lambda+yve_\lambda)h\varepsilon_i+\sum_{j>i}(xux_j+yvx_j)h\varepsilon_j.$$

Then we have $y\in N$, for $yve_\lambda h=0$, and hence $xve_\lambda h\varepsilon_{i+1}=xux_jh\varepsilon_{i+1}$. Therefore we obtain $x\in N$, for if x is a regular element of $e_\kappa Ae_\kappa$, $v\equiv ux_j$ mod. M' but from Lemma 1 this is impossible. Hence $Ae_\lambda\alpha\approx Ae_\lambda$ and the conclusion of Lemma 3 for this case is proved hereby.

Case (2). Next assume that Lemma 3 holds for $t-1$. Without losing of generality we shall assume that generators α_i of L are expressed as follows:

$$\alpha_i=e_\lambda h\varepsilon_{\nu_i}+\sum_{j\neq\nu_i}x_j^ie_\lambda h\varepsilon_j,$$

where $x_j^i\in e_\lambda Ne_\lambda$ for $j<\nu_i$, and $\nu_i<\nu_{i'}$ for $i<i'$. Then similarly as in case (1) the composition length of the intersection of $Ae_\lambda\alpha_1$ and $\sum_{j\geqq 2}Ae_\lambda\alpha_j$ is at most one. Hence the composition length of L is at least $(t-1)m$

3) $l(N)=\{x\in A\mid xN=0\}$, $r(N)=\{x\in A\mid Nx=0\}$.

$-(t-2)+m-1=tm-(t-1)$. Thus the proof of Lemma 3 is completed.

Now we shall return to the proof of Theorem 2. Assume that $H^{(n)}$ is decomposed into L_i, $i\leqq q$, as follows:

$$H^{(n)}=L_1\oplus\cdots\oplus L_q,\ q\geqq 2.$$

Let us denote by t_i the composition length of $L_i/NL_i\approx L_i+NH^{(n)}/NH^{(n)}$; then $\sum_i t_i=n$. It follows from Lemma 3 the composition length of $H^{(n)}$ is at least $\sum_{i=1}^{q}(t_im-(t_i-1))=nm-(n-q)$. But this is again impossible because the composition length of $H^{(n)}$ is $nm-(n-1)$, and hence $H^{(n)}$ is indecomposable. If we increase n, the composition length of $H^{(n)}$ $(=nm-(n-1))$ increases unlimitedly. This completes the proof of Theorem 2.

References

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