34. A Certain Type of Vector Field. II

By Toshiyuki MAEBASHI
Department of Mathematics, Hokkaido University

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All the notations of the previous paper [2] are included in the present paper.

I. Let C be a circle and let π be a symmetry, i.e. an idempotent isometry ($\neq$ the identity) leaving $O \in C$ fixed. Now consider an imbedding I of C into a 2-dimensional Euclid subspace E_2 of an n-dimensional Euclid space E_n. If O' is the other fixed point of π, denote by G the totality of Euclid motions g of E_n leaving both of $I(O)$ and $I(O')$ fixed. Then the orbit S by G of $I(C)$ is referred to as a *compact space of rotation*, if π keeps the curvature of the curve $I(C)$ invariant. Given a function $f(s)$ on C with $f \circ \pi(s) = f(s)$, we can extend it to one defined on the whole S in this way: Let $x \in S$ and $x = g(I(s))$ for $g \in G$ and $s \in C$. Then we set $f(s) = f(x)$. It is easy to see that $f(x)$ is well-defined.

Let $f_1(s)$ be such a function that $df_1 \neq 0$ except at O and O' and the above condition $f_1 \circ \pi = f_1$ hold. Then as is easily seen, the dual vector V_1 of *Grad* $[f_1(x)]$ satisfies 2) and 3) of Theorem A. Furthermore for s such that $V_1(I(s))$ is not proportional to $V_1(I \circ \pi(s))$, the vector field V satisfies 1) also at $x = g(I(s))$ for every $g \in G$. In fact there is a function f_2, in the neighborhood of such s, of the nature that the end point of the vector V_2[1] dual to *Grad* $[f_2(x)]$ remains fixed for the movement of $x \in S$ stated in the theorem. In addition, we can suppose that $f_2(s)$ has been chosen in such a fashion that π leaves $f_2(s)$ invariant and the domain of $f_2(s)$ is the set of all the s of the above-prescribed nature. For simplicity let us assume that the exceptional s are nowhere dense. Then V_1 and V_2 have the following properties respectively (we see these from Theorem A).

(1) *V_1 is a differentiable vector field defined on the whole S.*

(2) *The dual 1-form ω_1 to V_1 is closed.*

(3) *$A_{V_1} \in \mathfrak{P}^{-1}(\mathfrak{S}^*)$[2] except at $I(O)$ and $I(O')$.*

(1*) *V_2 is a differentiable vector field defined on a dense open*

1) Take a straight-line passing through $I(O)$ to the direction of $I(O')$ for the a-axis and introduce an orthogonal coordinate system in E_2. Then we have

$$\| V_2(I(s)) \| = \sqrt{1 + \left(\frac{da}{db}\right)^2}\, b$$

where a and b are the coordinates of $I(s)$.

2) For an exceptional s, we have $\mathfrak{P}(A_{V_1}) = 0$ at $x = g \circ I(s)$ $(g \in G)$.

subset K of S.

(2*) *The dual 1-form to V_2 is closed on K.*[3)]

(3*) $A_{V_2}\in\mathfrak{P}^{-1}(\mathfrak{S}^*)$ *at $x\in K$ except at $I(O)$ and $I(O')$.*

(4*) *For every $x\in K^c$ there is a function F defined in $U(x)\frown K$ and a gradient V_3 defined in $U(x)$ with $A_{V_3}\in\mathfrak{P}^{-1}(\mathfrak{S}^*)$ such that $FV_2=V_3$ in $U(x)\frown K$, where $U(x)$ means a neighborhood of x.*

If condition (3*) is replaced by the following (3**), we have the definition of the *torse-forming vector field in the large.*

(3**) $A_{V_2}\in\mathfrak{S}[P_{V_2}]$ *at K except at the 0-points of V that are assumed to form a nowhere dense subset of K.*

Conversely, if a Riemann or Finsler space admits a vector field satisfying (1*), (2*), (3*), (4*), and (iv′) and having a 0-point, then the space becomes a space of rotation, as has been shown implicitly in [1].

Although the singularity which V_2 possesses along K^c looks unessential for our theory, it is convenient to take such a singularity into consideration for the reason of its geometric meaning, an example of which is given by vector field (0.5) in the end of the introduction of [1].[4)]

II. In the rest of the present paper we shall prove the theorems announced in [2] by the use of Ricci calculus while developing a theory of pseudo-concurrent vector fields. The whole argument is only concerned with the local nature and all the vectors that appear below are assumed not to vanish. Taking a coordinate system, we write $\nabla^*_X\Omega=\Omega_{|i}X^i$ for a geometric object Ω. For details see the Cartan's textbook.

In the usual tensor notation the condition $A_V\in\mathfrak{P}^{-1}(\mathfrak{S}^*)$ turns out to be

$$V_{i|j}=A(x)g_{ij}+B(x)V_iV_j,\ \text{and}\ P_V^{5)}\circ Grad\,[A(x)]=Grad\,[A(x)],$$

where $A(x)$ is the function defined in **VI** of [2] (apart from the sign), which plays an important role in our theory.

III. *A pseudo-concurrent vector field V*, by definition, is one satisfying

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3) Since a closed form is locally integrable everywhere, the remark in p. 70 of [1] contains some erroneous statement. This mistake made the proofs of Theorem 4.7 more complicated than needed.

4) It is also true that the projective space came into the theory by the introduction of this singularity. In topological projective spaces, if W is an exceptional one stated in p. 82 of [1], Ω (see p. 86 there) is the inverse of a covering map and therefore not a 1–1 map. Then $\Omega(s)_\gamma$ should be considered as a continuous curve over γ.

5) The projection operator P_V acts rather on covector space T^*_x than on vector space T_x in the Finsler manifold. In fact, if $Z\in S_x$, then $||Z||$ depends only on x. Hence the operator P_Z which is defined by $P_Z(X^*)=X^*(Z)/||Z||$ $(X^*\in T^*_x)$ is one of T^*_x into itself.

(5) $\quad V_{i|j}=A^*(x)g_{ij}+B^*(x)V_iV_j+\rho_{|i}V_j+\rho_{|j}V_i$ and $V\in S_x$,

where $\rho_{|i}$ are the components of *Grad* $[\rho(x)]$. In what follows, we use the abbreviation "p. c." for pseudo-concurrent. The pseudo-concurrency of vector fields is an invariant concept under conformality, as will be shown later on.

The following lemma is easily obtained by a simple computation.

LEMMA. *A vector field V' satisfies* 1) *of Theorem A, if and only if the equation*: $V'_{i|j}=-g_{ij}+C(x)V'_iV'_j$ *with* $V'\in S_x$ *holds.*

After making some computation, we have

PROPOSITION 1. *A vector field V satisfies conditions* 1) *and* 2) *of Theorem A, if and only if V is a p. c. vector field given by* (5) *with this*:

(6) $$P_V\circ Grad\left[\frac{A^*(x)}{exp\,\rho}\right]=Grad\left[\frac{A^*(x)}{exp\,\rho}\right].$$

Let κ be the curvature vector of the tangent curve of a p. c. vector field (5). Then we have

PROPOSITION 2.

(7) $$\kappa=Grad\,\rho-P_V[Grad\,\rho].$$

COROLLARY. *A p. c. vector field becomes a torse-forming one if and only if the tangent curves are geodesics.*

If in the above corollary V satisfies (6), then the corresponding torse-forming vector field satisfies

(8) $$P_V\circ Grad\,[A(x)]=Grad\,[A(x)],$$

where $A(x)$ is defined in **VI** of [2].

These complete the proof of Theorem A.

REMARK. In proving Proposition 2, there is inevitable need for $\mathfrak{P}(A_V)$ not to equal 0 at the point concerned x. This is equivalent to saying that the direction indicated by V is not parallel at x. As a matter of fact, setting $Z=\frac{V}{\|V\|}$, we have $A_Z=0$ at x.

IV. The aim of this section is to make clear the geometric meaning of the p. c. vector field and the conformal map of the restricted kind.

A somewhat complicated computation is needed for the proofs of the following lemmas, but we can not go into details.

LEMMA 1. *For a p. c. vector field V, we have* $A_{ijk|l}V^kV^l=0$.

LEMMA 2. *Grad* ρ[6] $\in S_x$, *where* ρ *is the function having appeared in* (5).

From the latter we get

LEMMA 3. *For a p. c. vector field V*

(9) $$Grad\,\|V\|\in S_x.$$

6) We denote by *Grad* $F(x)$ the covector field the value at x of which is *Grad* $[F(x)]$.

On the other hand we can straightforwardly obtain the following theorem of the important nature.

THEOREM C. *A conformal map of the general kind transfers a p. c. vector field to another p. c. vector field.*

Using Lemma 2 and Theorem C we can find

THEOREM D. *A vector field V is p. c. if and only if it can be reduced to a torse-forming one by a conformal map of the restricted kind.*

V. Let R_{n-1} be an $(n-1)$-cell or an $(n-1)$-cubic and let L be an open interval. Then *a conformal family* is a diffeomorphism α of $R_{n-1}\times L$ into a Riemann or Finsler manifold M such that $\alpha(R_{n-1}\times\{t_1\})$ and $(R_{n-1}\times\{t_2\})$ are conformal (in the sense of the general kind) to each other in the natural way, where $t_1, t_2\in L$. We call the map T_z: $L\ni t\to\alpha(z,t)$ *a tangent curve of the conformal family.* Then the parameter t is referred to as *canonical.*

With this setting another meaning of the p. c. vector field is given by the following theorem, the proof of which is exactly the same as that of [3].

THEOREM E. *The existence of a p. c. vector field is equivalent to that of a conformal family such that the curvature vector of the tangent curves belongs to S_x at each point x.*

Besides there is the natural correspondence between p. c. vector fields and conformal families as well as in the case of torse-forming vector fields. Then the ratio of the metric form of $\alpha(R_{n-1}\times\{t_1\})$ and $\alpha(R_{n-1}\times\{t_2\})$ is given by the function $H(x)$ that is defined as follows. Let $x=\alpha(z, t_1)$. Then we set

$$H(x)=exp\int_{t_1}^{t_2}\frac{A\{T_z(t)\}}{\|V\|^2}dt. \tag{10}$$

Finally we would like to attract the readers' attention to the relation:

$$ds=\frac{dt}{\|V\|}, \tag{11}$$

where s denotes the arc length of T_z.

VI. Theorem B can be interpreted this way.

"*In order that there is a torse-forming vector field of one of these types:*

$$V_{i|j}=c(g_{ij}-V_iV_j) \tag{12}$$

$$V_{i|j}=c(g_{ij}+V_iV_j) \tag{13}$$

$$V_{i|j}=cg_{ij}, \tag{14}$$

it is necessary and sufficient that the metric form is conformally separable in one of the ways stated in Theorem B respectively."

In fact if $V'_{i|j}=c'g_{ij}+dV'_iV'_j$ for constants c and d, we have $V_{i|j}=c(g_{ij}\pm V_iV_j)$ by setting $V=\sqrt{\pm\frac{d}{c'}}\,V'$.

We shall leave the proof of the above theorem for the author's forthcoming paper.

References

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[3] M. Goto and S. Sasaki: Some theorems on holonomy groups of Riemannian manifolds, Trans. Amer. Math. Soc., **80**, 148-158 (1955).