49. Uniform Spaces with a U-extension Property

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In this paper, a function is real-valued and uniformly continuous, and a space is uniform. A space S has a *u-extension property* if a function defined on a uniform subspace of S has a uniform extension to S.

Recently, several mathematicians [2–8] have sought, and partly had, the conditions in order that a space has a u-extension property. In this paper, we shall find two necessary and sufficient conditions.

We shall say that a sequence of subsets is *U-discrete* for some entourage U in a space S if $U(x)$ meets at most one member of the sequence for every point $x\in S$, and call the following property of S the *property* $(*)$.

$(*)$ Let $\{A_n\}$ be a U-discrete sequence of subsets, and $\{a_n\}$ a sequence of natural numbers, then there is an entourage V with $V^{a_n}(A_n)\subset U(A_n)$ for every n.

Lemma. *A space S has the property $(*)$ if and only if, for any U-discrete sequence $\{A_n\}$ of subsets and a sequence $\{a_n\}$ of natural numbers, there is a function f on S with the value a_n on A_n and 0 on $S-\smile U(A_n)$.*

Proof. Suppose that S has the property $(*)$, then we have $V^{a_n}(A_n)\subset U(A_n)$ for some entourage V and every n. Using a fixed sequence $\{W_n\}$ of entourages with $W_0=V$, $W_{n+1}^2\subset W_n$, $W_n^{-1}=W_n$, we can make up a function f_n^i, for every n and i, with the value 1 on $V^i(A_n)$ and 0 on $S-V^{i+1}(A_n)$, $0\leq i\leq a_n-1$ (cf. [9], pp. 13–14).

$$\sum_{n=1}^{\infty}\sum_{i=0}^{a_n-1}f_n^i$$

is a desired function. The converse is clear if, for a given U in $(*)$, we take U_0 with $U_0^4\subset U$ instead of U in this lemma.

We remark that we may add "$0\leq f\leq a_n$ on $U(A_n)$" to the condition of f in this lemma.

Theorem. *A space S has the u-extension property if and only if S has the property $(*)$.*

Proof. Suppose that S has not the property $(*)$, then, by the Lemma, there are a U-discrete $\{A_n\}$ and a sequence $\{a_n\}$ of natural numbers which do not satisfy the condition in the Lemma, i.e., for W with $W^4\subset U$,

$$f(x)=\begin{cases}0 & \text{for } x\notin \cup W(A_n),\\ a_n & \text{for } x\in A_n,\end{cases}$$

has not any uniform extension to S, while f is uniformly continuous on $\cup A_n \cup (S-\cup W(A_n))$. Conversely, let S have the property (*), and f a given function defined on a uniform subspace A of S. Put $A_n = f^{-1}([n, n+1/3])$, $B_n = f^{-1}([n+1/3, n+2/3])$, and $C_n = f^{-1}([n+2/3, n+1])$. Then $\{D_{2n}=C_{2n-1}\cup A_{2n}\cup B_{2n};\ n=0, \pm 1, \cdots\}$ is V_1-discrete for some entourage V_1 with $V_1(D_{2n})\subset B_{2n-1}\cup D_{2n}\cup C_{2n}$, and, by the Lemma and its remark, there is a function f_1 on S with the value $2n$ on D_{2n} and 0 on $\cup A_{2n+1}$, and $0\le f_1\le 2n$ on $V_1(D_{2n})$. $\{E_{2n-1}=C_{2n-2}\cup A_{2n-1}\cup B_{2n-1};\ n=0, \pm 1, \cdots\}$ is V_2-discrete for some V_2 with $V_2(E_{2n-1})\subset B_{2n-2}\cup E_{2n-1}\cup C_{2n-1}$, and there is a function f_2 on S with the value $2n-1$ on E_{2n-1} and 0 on $\cup A_{2n-2}$, and $0\le f_2\le 2n-1$ on $V_2(E_{2n-1})$. $g_0=\max(f_1, f_2)$ is uniformly continuous on S and $|f(x)-g_0(x)|\le 2/3$ on A. Put $u=f-g_0$, $H=\{x\in A;\ -2/3\le u(x)\le -2/3^2\}$, and $K=\{x\in A;\ 2/3^2\le u(x)\le 2/3\}$, then, since u is uniformly continuous, there is a function v on S to $[-2/3^2, 2/3^2]$ with $v(H)=-2/3^2$, $v(K)=2/3^2$, and $|u(x)-v(x)|\le (2/3)^2$ on A. $g_1=g_0+v$ is uniformly continuous on S and $|f(x)-g_1(x)|\le (2/3)^2$ on A. In a similar way, we have a sequence of function g_n, and $\{g_n(x)\}$, $x\in S$, uniformly converges to $g(x)$, which coincides with $f(x)$ on A. g is uniformly continuous. In fact, for any positive number e, there is a natural number n with $|g(x)-g_n(x)|<e$ for every $x\in S$, there is a U such that $(x, y)\in U$ implies $|g_n(x)-g_n(y)|<e$, and thus we have

$$|g(x)-g(y)|\le |g(x)-g_n(x)|+|g_n(x)-g_n(y)|+|g_n(y)-g(y)|<3e$$

for every $(x, y)\in U$. Consequently, we have a uniform extension g of f to the whole space.

We say [1] a space is *uc* if any continuous real valued function is uniformly continuous.

Corollary 1. *A uc-space S has the u-extension property.*

Proof. Let $\{A_n\}$ be a U-discrete sequence of subsets, $\{a_n\}$ a sequence of natural numbers, and $V^3\subset U$. For every n, there is an entourage V_n with $V_n^{a_n}\subset V$. $\{A_n\}$ is discretely normally separated [1] by $\{V_n(A_n)\}$, so there is an entourage W such that $W(A_n)\subset V_n(A_n)$ for every n ([1], Theorem 1), and we have $W^{a_n}(A_n)\subset U(A_n)$ for all n.

Corollary 2. *A space S has the u-extension property if and only if its completion S^* also has.*

Proof. We shall prove only the "only if" part. Let $\{A_n\}$ be U-discrete in S^*, $\{a_n\}$ a sequence of natural numbers, and $U_0^4\subset U$. $\{U_0(A_n)\cap S\}$ is U_0-discrete in S. There is an entourage V_0 in S^* such that $V_0^{a_n}(U_0(A_n)\cap S)\subset U_0(U_0(A_n)\cap S)$ in S for every n. Suppose $V_1^4\subset V_0\cap U_0$, $x\in A_n$, and $y\in V_1^{a_n}(x)$, then there are points $x_0=x, x_1, \cdots, x_{a_n}=y$ in S^* with $(x_i, x_{i+1})\in V_1$, and, for every i, z_i in S with $(x_i, z_i)\in V_1$. Since $(z_i, z_{i+1})\in V_0$, we have $z_{a_n}\in V_0^{a_n}(z_0)$ in S, $z_0\in U_0(A_n)\cap S$, hence

$z_{a_n} \in U_0(U_0(A_n) \frown S) \subset U_0^2(A_n)$, and thus $y \in V_1 U_0^2(A_n) \subset U(A_n)$, i.e. $V_1^{a_n}(A_n) \subset U(A_n)$.

Added in proof. Using Katětov's theorem [8, Theorem 3], we can verify more simply the later half of the proof of the Theorem. In fact, since $u=f-g_0$ is bounded on A, u has a uniform extension w to S by Katětov's theorem, $w+g_0$ is a desired extension. The given form in the proof shows an elementary proof of Katětov's theorem.

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