62. *Harmonic Analysis on the Group of Linear Transformations of the Straight Line*

By Akihiro KOHARI

(Comm. by K. KUNUGI, M.J.A., May 13, 1961)

Introduction. Let G be the group of linear transformations of the straight line: $\xi \to g(\xi)=a\xi+b, (a>0, -\infty<b<\infty, -\infty<\xi<\infty)$. The irreducible unitary representations of G are described by I. M. Gelfand and M. A. Naimark (see the reference [1]). In the present paper, we shall investigate harmonic analysis on this group in more details and consider (1) spherical functions, (2) a generalization of the Laplace operator in the homogeneous space, (3) canonical bases of the representation spaces, (4) canonical Lie operators, and (5) an expansion formula by means of irreducible representations. We give the exact definitions of these notions in the following.

Let N be the subgroup $\{b \in G : b(\xi)=\xi+b\}$, and $\mathfrak{H}$ the Hilbert space of all complex-valued square integrable functions on the character group of N with respect to the usual Lebesgue measure. Then, for $f \in \mathfrak{H}$, the operators

$$U_g f(x)=\sqrt{a}\, e^{ibx} f(ax) \tag{0.1}$$

define a unitary representation of G. It is decomposed into the direct sum of two irreducible representations $\mathfrak{D}_\pm : \{U_g; \mathfrak{H}_\pm\}$, where the elements of $\mathfrak{H}_\pm$ are functions vanishing almost everywhere on the left or the right half line respectively. Any irreducible unitary representation is equivalent to either $\mathfrak{D}_+$, $\mathfrak{D}_-$ or a one dimensional representation of the subgroup $H=\{h: h(\xi)=a\xi\}$, [1]. We shall not consider the latter representations, for they have no significance to our problems, and except in 2, we shall consider only $\mathfrak{D}_+$ because all arguments are quite similar for $\mathfrak{D}_-$.

1. **Spherical functions.** For a moment let G be an arbitrary Lie group and H be a closed subgroup of G, and let $\mathfrak{D}: \{U_g; \mathfrak{H}\}$ be an irreducible unitary representation of G. If there exists a topological vector space $\mathfrak{F}$ which is dense in $\mathfrak{H}$ and a non-zero linear functional θ over $\mathfrak{F}$ such that

$$U_h^* \theta=\theta \quad \text{for all} \quad h \in H, \tag{1.1}$$

where U_h^* is the adjoint operator of U_h, then for $f \in \mathfrak{F}$, the function $\tilde{f}(g)=(f, U_g^* \theta)$ is constant on every right coset of G with respect to H. We shall call these functions *the spherical functions belonging to* $\mathfrak{D}$ (cf. [2]). In the present case, let $\mathfrak{F}_+$ be the space of all C^∞ functions whose carriers are compact and in the right half line, and

$\theta=1/\sqrt{x}$. Then $\mathfrak{F}_+$ and θ satisfy the condition (1.1), and the spherical functions belonging to $\mathfrak{D}_+$ are defined as follows:

$$\tilde{f}(b)=(f, U_g^*\theta)=\int_0^\infty e^{-ibx}f(x)/\sqrt{x}\,dx, \quad (-\infty<b<\infty) \tag{1.2}$$

where we can replace b by ξ since the homogeneous space G/H $(=N)$ is identified with the real axis $(-\infty<\xi<\infty)$.

Now we shall introduce some spaces of functions. Let $\Lambda_\pm$ be the set of such functions $\tilde{f}(\zeta)$ that are analytic on the upper (lower) half plane and uniformly square integrable along straight lines parallel to the real axis. Then the following three propositions hold (cf. [3]): (a) Each function of $\Lambda_\pm$ has the limit function on the real axis, in the sense both of almost everywhere convergence and of the limit in the mean. (b) A function is the limit function of an element of $\Lambda_\pm$, if and only if its inverse Fourier transform vanishes almost everywhere on the left (or right) half line. (Let us denote the set of such functions by $\mathrm{M}_\pm$.) (c) The inverse of the limiting process in (b) is given, for $\tilde{f}(\xi)$ of $\mathrm{M}_\pm$, by the Cauchy integral:

$$\tilde{f}(\zeta)=\pm 1/2\pi i\int_{-\infty}^\infty \tilde{f}(\xi)/(\zeta-\xi)d\xi. \tag{1.3}$$

Because (1.2) is nothing but the usual Fourier transformation of a function $f(x)/\sqrt{x}$ vanishing almost everywhere on the left half of the real axis, the space of spherical functions belonging to $\mathfrak{D}_+$ is dense in M_+ (or of Λ_+). We denote this subspace by Φ_+, and moreover put $\Phi=\Phi_+\oplus\Phi_-$. Then Φ is a subspace of $\Lambda\equiv\Lambda_+\oplus\Lambda_-$ and is dense in $L^2(N, db)$, where $N=G/H$. $\mathfrak{D}$ is transformed into the representation:

$$T_g\tilde{f}(\xi)=\sqrt{a}\,\tilde{f}(a\xi+b) \tag{1.4}$$

in the space Φ. It is irreducible on the closure of $\Phi_\pm$.

2. Invariant operator. Let a Lie group G and a closed subgroup H be given. For them we shall generalize the notion of the Laplace operator defined in [4], and call an operator which is defined in a space of functions on the homogeneous manifold G/H and commutes with all transformations $T_g f(x)=f(g^{-1}x)$ *invariant operator* on G/H. In our case, it is obtained as follows.

Put $Y(x)=\pm 1$, according to $x>0$ or $x<0$. Then, the multiplication operator by $Y(x)$ on the space $\mathfrak{F}=\mathfrak{F}_+\oplus\mathfrak{F}_-$ has eigenspaces $\mathfrak{F}_+$ and $\mathfrak{F}_-$ corresponding to the eigenvalues $\pm 1: Y(x)f(x)=\pm f(x)$, for $f\in\mathfrak{F}_\pm$. Multiplying $1/\sqrt{x}$ to this equation consider the Fourier transform. Then we obtain in the space Φ symbolically

$$\tilde{Y}*\tilde{f}(\xi)=\pm\tilde{f}(\xi) \quad \text{for} \quad \tilde{f}\in\Phi. \tag{2.1}$$

This holds not only in the symbolical sense but as the functional equation in Φ. In fact the Fourier transform of Heaviside function $Y(x)$ is $\tilde{Y}(\xi)=\text{v.p.}(1/2\pi i\xi)$ (cf. [5]). If we consider $\tilde{Y}*\tilde{f}$ on the complex plane, that is, in the space Λ, we get

$$(\tilde{Y}*\tilde{f})(\zeta)=1/2\pi i\int_{-\infty}^{\infty}\tilde{f}(\xi)/(\zeta-\xi)d\xi.$$

For $\tilde{f}\in\Phi_{\pm}$, this function has the limit function on the real axis which is equal to $\pm\tilde{f}(\xi)$, by the propositions (a) and (b). Conversely, a function $\tilde{f}\in\Phi$ that satisfies (2.1) belongs to Φ_{+} or Φ_{-} according to the eigenvalue $+1$ or -1. Thus each space of spherical functions belonging to $\mathfrak{D}_{\pm}$ is the eigenspace of the operator $\tilde{Y}*$ on Φ. Moreover it is easy to show that the operator $\tilde{Y}*$ in the space Φ commutes with every transformation $T_g: T_g f(\xi)=f(a\xi+b)$. Thus the convolution operator $\tilde{Y}*$ is the invariant operator on G/H in our sense. We remark that $\tilde{Y}*$ is defined on $L^2(N)$.

3. **Canonical bases.** The functions $x^{i\lambda}$ $(-\infty<\lambda<\infty)$, defined on the right half of the real axis, are regarded as the canonical basis in $\mathfrak{H}_{+}$, in the sense that they are invariant with respect to U_h for $h\in H$ and the Parseval equality holds in the expansion formula by means of $x^{i\lambda}$. Now, for the space Φ_{+} of spherical functions, we shall define the canonical basis as follows:

$$\chi_{\lambda}^{+}(\xi)\equiv\chi_{\lambda}^{+}(b)\equiv(x^{i\lambda},\ U_g^{*}\theta)=\int_{0}^{\infty}e^{-ibx}x^{i\lambda-1/2}dx. \tag{3.1}$$

It is well known that the formula

$$\int_{0}^{\infty}e^{-\beta x}x^{\alpha-1}dx=\beta^{-\alpha}\Gamma(\alpha)$$

is valid for $\Re(\alpha)$ and $\Re(\beta)>0$, but we can prove it also for $\Re(\beta)=0$ and $0<\Re(\alpha)<1$. The computation is rather complicated and we omit it. By this and (3.1), $\chi_{\lambda}^{+}(\xi)=(i\xi)^{-i\lambda-1/2}\Gamma(1/2+i\lambda)$. Hence, setting $\alpha=1/2-i\lambda$, we obtain

$$\chi_{\lambda}^{+}(\xi)=(i)^{\alpha-1}\pi/\sin(\pi\alpha)\times\xi^{\alpha-1}/\Gamma(\alpha).$$

It is usual to write (cf. [5]) $Y_{\alpha}(\xi)=\text{p.f.}(\xi^{\alpha-1}/\Gamma(\alpha))$ for $\xi>0$, and $=0$ for $\xi<0$, where the symbol p.f. is unnecessary for $\Re(\alpha)>0$. Hence

$$\chi_{\lambda}^{+}(\xi)=K_{+}(\alpha)Y_{\alpha}(\xi)+K_{-}(\alpha)Y_{\alpha}(-\xi), \tag{3.2}$$

where $K_{\pm}(\alpha)=(\pm i)^{\alpha-1}\pi/\sin(\pi\alpha)$.

For the representation $\mathfrak{D}_{-}$, we define the canonical basis by

$$\chi_{\lambda}^{-}(\xi)=((-x)^{i\lambda},\ U_g^{*}\theta),$$

and it is showed quite similarly that

$$\chi_{\lambda}^{-}(\xi)=K_{-}(\alpha)Y_{\alpha}(\xi)+K_{+}(\alpha)Y_{\alpha}(-\xi).$$

The functions $\chi_{\lambda}^{+}(\xi)$ are eigenfunctions of operators T_h: From (3.2) and (1.4) we have

$$T_h\chi_{\lambda}^{+}(\xi)=a^{\alpha-1}\chi_{\lambda}^{+}(\xi),\qquad \alpha=1/2-i\lambda. \tag{3.3}$$

4. The canonical Lie operators. Let X_1 and X_2 be the bases of the Lie algebra of G which generate the subgroups H and N respectively:

$$X_1=\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \qquad X_2=\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$$

Then

$$[X_1, X_2]=X_2. \tag{4.1}$$

We use the same notations for the infinitesimal operators on Φ corresponding to X_i, that is

$$\begin{aligned} X_1\tilde{f}&\equiv\lim(T_{a+\Delta a}\tilde{f}-T_a\tilde{f})/\Delta a\,|_{a=1}=b\, df/db=b\, df/d\xi, \\ X_2\tilde{f}&\equiv\lim(T_{b+\Delta b}\tilde{f}-T_b\tilde{f})/\Delta b\,|_{b=0}=df/db=df/d\xi. \end{aligned} \tag{4.2}$$

They are defined on the space Φ and (4.1) holds also for them. We remark that if $X_1\tilde{f}=\alpha\tilde{f}$, then from the relation

$$X_1(X_2\tilde{f})=X_2\tilde{f}+X_2(X_1\tilde{f})=(\alpha+1)(X_2\tilde{f}),$$

we have $X_1(X_2^n\tilde{f})=(\alpha+n)(X_2^n\tilde{f})$.

Now, let us consider the adjoint operators of X_i on the adjoint space Φ^* that contains the canonical bases $\chi_\lambda^\pm$. We obtain from (3.3)

$$X_1^*(\chi_\lambda^+)=(1/2-i\lambda)\chi_\lambda^+.$$

It means that χ_λ^+ are eigenfunctions of X_1^*. From this and the above argument, it is natural to regard the formal operators $(X_2^*)^{i\mu}(-\infty<\mu<\infty)$ as the canonical operators. Here $(X_2^*)^{i\mu}$ are defined as follows. Generally a differential operator of non-integral (complex) order is defined as the convolution operator of the monomial pseudo-function Y_τ (cf. [5]), that is, $(d/d\xi)^\tau f=Y_{-\tau}*f$. Thus, from (4.2), we have

$$(X_2^*)^\tau\tilde{f}=Y_{-\tau}*\tilde{f}, \qquad \text{for } \tilde{f}\in\Phi.$$

The operator $(X_2^*)^{i\mu}$ transforms χ_λ^+ to $\chi_{\lambda+\mu}^+$ up to a constant factor. In fact, using (3.2) and the well-known formula $Y_\sigma*Y_\tau=Y_{\sigma+\tau}$, we have

$$(X_2^*)^{i\mu}\chi_\lambda^+(\xi)=C(\lambda,\mu)\chi_{\lambda+\mu}^+(\xi),$$

where $C(\lambda,\mu)=\exp(-\pi\mu/2)\sin\pi(\alpha-i\mu)/\sin\pi\alpha$.

5. The expansion formula. We consider here the expansion of Φ by means of $\chi_\lambda^\pm$. For $\tilde{f}_+\in\Phi_+$, put

$$\hat{f}_+(\lambda)=(\tilde{f}_+,\chi_\lambda^+)=\int_{-\infty}^{\infty}\tilde{f}_+(\xi)\,\overline{\chi_\lambda^+(\xi)}d\xi, \qquad (-\infty<\lambda<\infty). \tag{5.1}$$

Let us denote the set of functions $\hat{f}_+(\lambda)$ by Ψ_+, and define the mapping $\mathcal{F}$ from Φ_+ onto Ψ_+, by superposing the mapping $\mathcal{F}_1$ from $\mathfrak{F}_+$ to Φ_+ and $\mathcal{F}_2$ from $\mathfrak{F}_+$ to Ψ_+, which is defined below.

The mapping $\mathcal{F}_1$ was already considered in 1 and, because (1.2) is the usual Fourier transformation of $f_+(x)/\sqrt{x}$, there hold the inversion formula and the Parseval equality:

$$f_+(x)/\sqrt{x}=\int_{-\infty}^{\infty}e^{i\xi x}\tilde{f}_+(\xi)d\xi, \tag{5.2}$$

$$\int_0^\infty |f_+(x)|^2\, dx/x = \int_{-\infty}^\infty |\tilde{f}_+(\xi)|^2\, d\xi. \tag{5.3}$$

The mapping $\mathcal{F}_2$ from $\mathfrak{F}_+$ to Ψ_+ is given as follows:

$$\hat{\varphi}_+(\lambda) = (f_+, x^{i\lambda}) = \int_0^\infty x^{-i\lambda} f_+(x)\, dx/x, \tag{5.4}$$

for $f_+(x)$ of $\mathfrak{F}_+$. Since it is the usual Fourier transformation (set $x=e^s$) we have

$$f_+(x) = \int_{-\infty}^\infty x^{i\lambda} \hat{\varphi}_+(\lambda)\, d\lambda, \tag{5.5}$$

$$\int_0^\infty |f_+(x)|^2\, dx/x = \int_{-\infty}^\infty |\hat{\varphi}_+(\lambda)|^2\, d\lambda. \tag{5.6}$$

On the other hand by (5.2), (5.4) and (5.1)

$$\hat{\varphi}_+(\lambda) = \hat{f}_+(\lambda). \tag{5.7}$$

Hence (5.4) and (5.5) give a one to one and onto correspondence between Φ_+ and Ψ_+. Therefore we have $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1^{-1}$ or by inserting (5.5) into (1.2) we obtain

$$\tilde{f}_+(\xi) = \int_{-\infty}^\infty \hat{f}_+(\lambda) \chi_\lambda^+(\xi)\, d\xi,$$

and from (5.3), (5.4) and (5.7)

$$\int_{-\infty}^\infty |\hat{f}_+(\lambda)|^2\, d\lambda = \int_{-\infty}^\infty |\tilde{f}_+(\xi)|^2\, d\xi.$$

Now, if we take $\tilde{f}$ from Φ, then since $\tilde{f} = \tilde{f}_+ + \tilde{f}_-$, and all the above arguments hold quite analogously for $\tilde{f}_-$, the expansion formula is written as

$$\tilde{f}(\xi) = \int_{-\infty}^\infty \hat{f}_+(\lambda)\, \chi_\lambda^+(\xi)\, d\lambda + \int_{-\infty}^\infty \hat{f}_-(\lambda)\, \chi_\lambda^-(\xi) d\lambda.$$

Moreover, since Φ_+ and Φ_- are orthogonal in $L^2(N, db)$, we have

$$\|\tilde{f}\|^2 = \|\hat{f}_+\|^2 + \|\hat{f}_-\|^2,$$

which means, in a sense, orthogonality of the canonical bases $\chi_\lambda^\pm$ in Φ.

References

[1] I. M. Gelfand and M. A. Naimark: Unitary representations of the group of linear transformations of the straight line, Dokl. Akad. Nauk, **55**, 567–570 (1947).
[2] H. Yoshizawa: Representations of groups and spherical functions (in Jap.), Sugaku, **12**, fasc. 1 (1960).
[3] E. Hille and J. D. Tamarkin: On the absolute integrability of Fourier transformation, Fund. Math., **25**, 329–352 (1935).
[4] I. M. Gelfand: Spherical functions in symmetric Riemannian spaces (in Russian), Dokl. Akad. Nauk, **70**, 5–8 (1950).
[5] L. Schwartz: Théorie des Distribution, I, Hermann, Paris (1950).