53. *A Remarkable Divergent Fourier Series*

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It is very well known that A. N. Kolmogorov [2] was the first to construct an example of a function $f(x)\in L(0,2\pi)$ whose Fourier-Lebesgue trigonometric series diverges almost everywhere. Later he constructed a Fourier-Lebesgue series which diverges unboundedly everywhere [3]. But the Fourier series given by Kolmogorov is not a Fourier series of a function $f(x)\in L\log^+ L$, since its conjugate series is not a Fourier series.[1] The next step forward was made by G. H. Hardy and W. W. Rogosinski [1]. They constructed an almost everywhere divergent Fourier series whose conjugate series is also a Fourier series.[2]

In another direction, K. Zeller [8] gave a method to construct a Fourier series which converges on an arbitrary set $E\subset(0,2\pi)$ of the type F_σ (denumerable sum of closed sets) and diverges unboundedly on $E_1=[0,2\pi)-E$. Recently L. V. Taikov [6] constructed a Fourier series which converges on $E\subset[0,2\pi)$ of the type F_σ and diverges unboundedly everywhere on $E_1=[0,2\pi)-E$ such that the conjugate series is also a Fourier series.

It is natural to inquire whether the Fourier series of a function $f(x)$ belonging to $L^2(0,2\pi)$ converges almost everywhere. This was conjectured by N. N. Lusin in the positive sense some forty-five years ago,[3] but it has neither been proved nor been disproved. To attack this difficult problem, it is of interest to observe the maximum speed at which a Fourier series may diverge unboundedly almost everywhere. If there exists a Fourier series which diverges very fast, we might think that the Lusin's conjecture could not be true. Concerning to this point, A. Zygmund ([10], p. 308) conjectured that for any sequence of positive numbers $\lambda_n=o(\log n)$, $n\to\infty$, there is an $f\in L$ such that at almost every point x we have $S_n(x;f)>\lambda_n$ for infinitely many n, where $S_n(x;f)$ denotes the nth partial sum of the Fourier

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1) See, for example, [10] p. 308 and [7] Theorem 9. But, the series considered in [7] §3 is different from the original series defined by Kolmogorov, since the function $\phi_n(x)$ defined in [7] §3 is not a Féjer kernel. Each function $f(x)$ of the class denoted by $L\log^+ L$ is such that $|f(x)|\log^+|f(x)|\in L(0,2\pi)$.

2) In the English translation of [6]: Soviet Math., **6**, No. 2, p. 347, it is stated that Hardy and Rogosinski constructed an everywhere divergent Fourier series whose conjugate series is also a Fourier series, but it has been wrongly translated, cf. also [5].

3) See [4] p. 219.

series of $f(x)$. The purpose of this paper is to present a function $f(x)$ whose Fourier-Lebesgue trigonometric series diverges unboundedly at every point with the scale of $o(\log\log n)$, such that the conjugate series of the Fourier of $f(x)$ is also a Fourier series. From this result it is natural to conjecture that there exists a function $f(x)\in L(0,2\pi)$ with $\bar{f}(x)\in L(0,2\pi)$ and $|f(x)|\log^{+}\log^{+}|f(x)|\in L(0,2\pi)$ such that the Fourier series of $f(x)$ diverges everywhere in $[0,2\pi)$.

LEMMA 1. Let $M>0$. Then for each trigonometric polynomial

$$(1)\qquad T(x)=\frac{a_0}{2}+\sum_{k=1}^{N}a_k\cos kx+b_k\sin kx,$$

there exists a trigonometric polynomial of the form

$$(2)\qquad t(x)=\sum_{k=Q}^{R}c_k\cos kx+d_k\sin kx,$$

where $Q>M$, such that, for each $x\in[0,2\pi)$,

$$(3)\qquad |t(x)|\leqq|T(x)|,\quad |\bar{t}(x)|\leqq|T(x)|,$$

$$(4)\qquad \frac{1}{8}\sup_k|S_k(x;\,T)|\leq\sup_n|S_n(x;\,\bar{t})|\leq\sup_k\{|S_k(x;\,t)|+|S_k(x;\overline{T})|\},$$

$$(5)\qquad \frac{1}{8}\sup_k|S_k(x;\,T)|\leq\sup_n|S_n(x;\,t)|,$$

where $\bar{t}(x)$ and $\overline{T}(x)$ are respectively the conjugate functions of $f(x)$ and $T(x)$.
Lemma 1 is due to L. V. Taikov [6].[4] In what follows we shall denote, by $K_1, K_2,\cdots$, some positive constants.

LEMMA 2. There is a sequence of non-negative trigonometric polynomials $F_1, F_2,\cdots, F_n,\cdots$ of orders $\nu_1<\nu_2<\cdots$, with constant term 1 and having the following properties. With each n we can associate a number $A_n=K_1\log n$, a set $E_n=[0,4\pi(n-\sqrt{n})/(2n+1)]\subset[0,2\pi)$, and an integer λ_n such that

(i) $\lambda_n\nearrow\infty$;

(ii) for each $x\in E_n$, there is an integer k satisfying $\lambda_n\leq k\leq\nu_n$, $20\,n<k=k(x_1n)<20\,K_2^{n-\sqrt{n}}n^{2(n-\sqrt{n})+1}$ for sufficiently large n, and such that

$$(6)\qquad S_k(x;\,F_n)>A_n=K_1\log n>K_3\log\log k,$$

for sufficiently large n.

It is sufficient to prove (6) and to estimate the value of k; and we omit further details of the proof which have been given in [10] pp. 310–311. We now follow the details in [9] pp. 175–179, in which the method is different from the argument given in [10] p. 313

4) See Lemma 2 in [6]. There is a slip in p. 784, where $|\cos px|\leqq\frac{1}{2}$ should read $|\cos px|\leqq\frac{1}{4}$ and we have to replace $\frac{1}{4}\sup_k|S_k(x:T)|$ in formula (2) of [6] by $\frac{1}{8}\sup_k|S_k(x:t)|$ in our formula (4). Our formula (5) has been established in [6], p. 784.

which depends on the theory of distribution. In the first place, we need to give a precise estimate of the value of $\delta=\delta_n$ which is defined in [9], p. 176. Writing $K_m(t)$ for the mth Féjer kernel of t:

$$K_m(t)=\frac{1}{2(m+1)}\left\{\frac{\sin\left(\frac{m+1}{2}\right)t}{\sin\frac{1}{2}t}\right\}^2, \tag{7}$$

we define, as in [9] p. 176,

$$F_n(x)=\phi(x)+\psi(x)=K_m\{(2n+1)x\}+\frac{1}{n+1}\sum_{i=0}^{n}K_{m_i}(x-x_{2i}), \tag{8}$$

where $x_i=2\pi i/(2n+1)$, $M\leqq m_0<m_1<\cdots$, and the numbers m_j will be defined later. We now set $\phi(x)=K_m\{(2n+1)x\}\geqq n$, for $x\in I_i=(x_i-\delta, x_i+\delta)$, $i=1,2,\cdots,2n$. Taking $m=m_0=M=20n$, $t=(2n+1)x$, we have

$$\begin{cases} K_m(t)=\frac{1}{2(m+1)}\left\{\frac{\sin\left(\frac{m+1}{2}\right)t}{\sin\frac{1}{2}t}\right\}^2\geqq\frac{2}{(m+1)t^2}\left\{\sin\left(\frac{m+1}{2}\right)t\right\}^2 \\ \qquad\geqq\frac{2}{(m+1)t^2}\left\{\frac{(m+1)}{2\pi}t\right\}^2=\frac{m+1}{2\pi^2}>\frac{m+1}{20}>n, \end{cases} \tag{9}$$

for $0<\frac{(m+1)t}{2}<\frac{\pi}{2}$ and $0<x<\frac{\pi}{20n(n+1)}$. So it is sufficient to take $\delta=\frac{1}{20n^2}$, for sufficiently large n. We now write, as in [9] p. 178, $x_{2j+2}-x=4\pi\theta/(2n+1)$. Then $x\in I'_{2j}+I'_{2j+1}$ if and only if $\theta\in\left\{\left(\eta,\frac{1}{2}-\eta\right)+\left(\frac{1}{2}+\eta,1-\eta\right)\right\}$, where $I'_j=(x_j+\delta,x_{j+1}-\delta)$, $j=0,\cdots,2n$. This means $\eta=\frac{\delta}{4\pi}=\frac{1}{80\pi n^2}>\frac{1}{320n^2}$. We proceed to estimate the values of m_j, $0\leqq j\leqq n-\sqrt{n}$, which are defined in [9] pp. 178–179. This corresponds to the following cases:

(a) $2\theta\in\left(2\eta,\frac{1}{3}\right)$, (b) $2\theta\in\left(\frac{1}{3},\frac{2}{3}\right)$, (c) $2\theta\in\left(\frac{2}{3},1-2\eta\right)$.

From cases (a) and (c), we obtain

$$m'_j\leq m_j+\frac{2(1/12)}{2(1/320n^2)}=m_j+\frac{80}{3}n^2. \tag{10}$$

The case (b) can be decomposed into:

(α) $2\theta\in\left(\frac{1}{3},\frac{5}{12}\right)$, ($\beta$) $2\theta\in\left(\frac{5}{12},\frac{7}{12}\right)$, ($\gamma$) $2\theta\in\left(\frac{7}{12},\frac{2}{3}\right)$.

In cases (α) and (γ), 4θ belongs to either $\left(\frac{2}{3},\frac{5}{6}\right)$ or $\left(\frac{1}{6},\frac{1}{3}\right)$, and this gives

$$m'_j\leq m_j+\frac{160}{3}n^2. \tag{11}$$

It remains to consider the case (β). Following the argument in [9] p. 179, we have

$$(12)\qquad m'_j \le m_j + \frac{320}{3} n^2 m_j = \left(1 + \frac{320}{3} n^2\right) m_j < K_4 n^2 m_j.$$

Since we may take $m_{j+1} = 2m'_j + 1$, and therefore we have the estimate:

$$(13)\qquad m_j \le K_5^j n^{2j} m_0 \le K_5^{n-\sqrt{n}} n^{2(n-\sqrt{n})} \cdot 20n,\ j=0, 1, \cdots, [n-\sqrt{n}\,].$$

If $x \in (I'_{2j} + I'_{2j+1})$, then the value $k = k(x, n)$ is defined by $m_j \leqq k < \frac{1}{2} m_{j+1}$. This means $\log\log k < K_6 \log n$, for sufficiently large n, and therefore we obtain

$$(14)\qquad S_k(F_n; x) > K_7 \log n > K_8 \log\log k,\ x \in \sum_{j=0}^{[n-\sqrt{n}]} (I'_{2j} + I'_{2j+1}),$$

$$(15)\qquad S_k(F_n; x) = S_M(F_n; x) = S_{20n}(F_n; x) \geqq \frac{1}{2} n,\ x \in \sum_{i=0}^{2n} I_i.$$

The inequality (6) then follows immediately from (14) and (15).

THEOREM 1. Given any sequence of positive numbers $p_n = o(\log\log n)$, $n \to \infty$, there exists a function $f(x)$ with conjugate series being a Fourier series, such that at every point x, $S_n(x; f) > p_n$ for infinitely many n.

Proof. We first take a trigonometric polynomial $t_n(x)$ obtained from $F_n(x)$ as in Lemma 1, so that there is no overlapping of terms occured in the following trigonometric series:

$$(16)\qquad \sum_1^\infty \frac{t_n(x)}{B_n} = \sum_{n=1}^\infty \frac{1}{B_n} \sum_{j=P(n)}^{[K_2^{n-\sqrt{n}} 20 n^{2(n-\sqrt{n})+1} + P(n)]} (a_j \cos jx + b_j \sin jx),$$

where the constants B_n, a_j, b_j will be defined later. This means

$$(17)\qquad 20 K_2^{n-\sqrt{n}} n^{2(n-\sqrt{n})+1} + P(n) < P(n+1),$$

$$(18)\qquad P(n+1) - P(n) > K_9^{n-\sqrt{n}} n^{2(n-\sqrt{n})+1}.$$

It is sufficient to take

$$(19)\qquad \begin{cases} P(n) \simeq \displaystyle\int_1^n K_9^{x-\sqrt{x}} x^{2(x-\sqrt{x})+1} dx \\ \qquad < \displaystyle\int_1^n e^{2x^2} x\, dx < e^{2n^2} \quad (n \ge N), \end{cases}$$

for sufficiently large n. So we may take $P(n)$ equals to the integral part of e^{2n^2}: $P(n) = [e^{2n^2}]$. We may assume, without loss of generality, that $p_n / \log\log n$ decreases steadily to zero. Then we set B_n such that

$$(20)\qquad \frac{1}{B_n} > \frac{16 p_k}{K_3 \log\log k},$$

for all values of $k = k(x, n)$ such that

$$(21)\qquad 20n + P(n) < k = k(x, n) < P(n) + 20 K_2^{n-\sqrt{n}} n^{2(n-\sqrt{n})+1}.$$

It follows that

(22) $$\frac{16p_{[20n+P(n)]}}{K_3 \log\log\{20n+P(n)\}} < \frac{1}{B_n}.$$

It is then suffiicient to take

(23) $$B_n = \frac{K_3 \log\log 20n}{16p_{20n}},$$

which increases monotonically to infinity, as $n\to\infty$.
Next, let us define n_i and

(24) $$f(x) = \sum_{i=0}^{\infty} t_{n_i}(x)/B_{n_i},$$

so that $\sum_{i=1}^{\infty} 1/B_{n_i} < \infty$. From Lemma 1 and Lemma 2, we see that $f(x)\in L(0, 2\pi)$, and the series (24) has infinitely many blocks of non-overlapping trigonometric polynomials, such that

(25) $$\begin{cases} |S_k(x; f)| > \dfrac{1}{2}\dfrac{S_k(x; t_n)}{B_n} > \dfrac{1}{2}\cdot\dfrac{1}{8}\dfrac{S_k(x; T_n)}{B_n} \\ \qquad > \dfrac{1}{16}K_3 \log\log k\cdot\dfrac{16p_k}{K_3\log\log k} = p_k, \end{cases}$$

for infinitely may k satisfying

(26) $$k > P(n_i) = [e^{2n_i^2}].$$

It remains to show that $\bar{f}(x)\in L(0, 2\pi)$. By Lemma 1, it follows that

(27) $$\begin{cases} \displaystyle\int_0^{2\pi} |\bar{f}(x)|\,dx \leqq \sum_{i=0}^{\infty}\int_0^{2\pi} |\bar{t}_{n_i}(x)|\,dx/B_{n_i} \\ \qquad \displaystyle\leqq \sum_{i=0}^{\infty}\frac{1}{B_{n_i}}\int_0^{2\pi}|F_{n_i}(x)|\,dx = 2\pi\sum_{i=0}^{\infty}\frac{1}{B_{n_i}} < \infty. \end{cases}$$

Hence $\bar{f}(x)\in L(0, 2\pi)$. This completes the proof of Theorem 1. The following theorem is a direct consequence of above Theorem 1 (cf. also [5]):

THEOREM 2. Given any sequence of positive numbers $p_n = o(\log\log n)$, $n\to\infty$, there exists a Fourier series belonging to the class H, such that at every point $x\in[0, 2\pi)$, $S_n(x, f) > p_n$ for infinitely many n.

Added in Proof. The author is indebted to Prof. P. L. Ul'yanov for pointing out a mistake during the preparation of this paper.

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