67. On a Ya. B. Rutickii's Theorem Concerning a Property of the Orlicz Norm

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Let $\boldsymbol{R}$ be a modulared semi-ordered linear space[1] and its modular be $m(x)(x\in\boldsymbol{R})$, and suppose the semi-regularity[2] of $\boldsymbol{R}$.

Concerning the property of the Orlicz norm:

$$(1)\qquad \lim_{\|u\|_M\to\infty}\frac{1}{\|u\|_M}\int_G M[|u(t)|]dt=\infty,$$

where G is a bounded closed set in finite-dimentional Euclidian space and M is a N-function (see [2]), Ya. B. Rutickii [4] gave the following theorem.

Theorem 1. *In order that* (1) *be fulfilled, it is necessary and sufficient that there exists a function* $f(u)$ $(0\leqq u<\infty)$, *satisfying the condition*

$$(2)\qquad \lim_{u\to\infty} f(u)=\infty$$

and such that for every v and all sufficiently large values of u the inequality

$$(3)\qquad M(uv)\geqq uf(u)M(v)$$

be fulfilled.

The Orlicz space $\boldsymbol{L}_M^*$ is a modulared space on which the modular is defined as

$$(4)\qquad m(x)=\int_G M[|x(t)|]dt \qquad (x\in\boldsymbol{L}_M^*).$$

Then, (1) is written as

$$(5)\qquad \lim_{\|x\|\to\infty} m(x)/\|x\|=\infty$$

where $$\|x\|=\inf_{\xi>0}\frac{1+m(\xi x)}{\xi}\ (=\|x\|_M)^{3)}$$

The purpose of this paper is to prove the following theorem.

Theorem 2. *Let $\boldsymbol{R}^m$ be a modulared semi-ordered linear space. Then, in order that* (5) *be fulfilled, it is necessary and sufficient*

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1) Namely, $\boldsymbol{R}$ is a conditionally vector lattice, in the sense of G. Birkhoff, on which a functional $m(x)$ is defined, and then such space is denoted by $\boldsymbol{R}^m$. (see [3, §35]).

2) $\boldsymbol{R}$ is said to be *semi-regular*, if for any $o\neq x\in\boldsymbol{R}$ there exists an element $\bar{x}\in\overline{\boldsymbol{R}}$ such that $\bar{x}(x)\neq 0$, where $\overline{\boldsymbol{R}}$ is the totality of all linear functionals $\bar{x}$ satisfying that $x_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\inf_{\lambda\in\Lambda}|\bar{x}(x_\lambda)|=0$.

3) See Theorem 10.5 in [2].

that the conjugate modular[4] $\overline{m}$ *of* m *is uniformly finite.*[5]

Before proceeding to the proof, we will observe a relation between Theorems 1 and 2.

The conjugate modular $\overline{m}$ of the modular m which is defined by (4) is given as

$$\overline{m}(\overline{x})=\int_G N[|\overline{x}(t)|]dt \qquad (\overline{x}\in \boldsymbol{L}_N^*),$$

where N is the complementary function of M in the sense of Young.

This modular $\overline{m}$ is uniformly finite, if and only if N satisfies the $(\varDelta_2)$-*condition*, i.e., there exist constant numbers K and $v_0>0$ such that

$$N(2v)\leqq KN(v) \qquad (v_0\leqq v). \tag{6}$$

Therefore, by Theorem 2, *we can set* (6) *instead of* (2) *and* (3), *in Rutickii's theorem.*

To prove Theorem 2, we restate the following results concerning the modular.

Lemma 1. *If a modular* $m(x)$ $(x\in R^m)$ *is uniformly increasing,*[6] *then the conjugate modular* $\overline{m}$ *of* m *is uniformly finite. Conversely, if* $\overline{m}$ *is uniformly finite, then* m *is uniformly increasing.*

The first statement of Lemma is Theorem 48.4 in [3], and the next is reduced from the reflexivity[7] of m and the uniform increaseness of the conjugate modular $\overline{\overline{m}}$ of $\overline{m}$.

Lemma 2. (*Lemma* 3.1 in [1]) *For* $a\in \boldsymbol{R}^m$, $m(a)<|||a|||$[8]$=1$, *if and only if* $m(a)<1$ *and* $m(\xi a)=\infty$ *for all* $\xi>1$.

The proof of Theorem 2. Suppose that $m(x)$ fulfills (5). Then, from the equivalence between $\|\cdot\|$ and $|||\cdot|||$, we have

$$\lim_{|||x|||\to\infty} m(x)/|||x|||=\infty.$$

If m is not uniformly increasing, then there exists a constant number C such that

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4) The *conjugate modular* $\overline{m}$ of m is to be defined as

$$\overline{m}(\overline{a})=\sup_{x\in \boldsymbol{R}^m}\{\overline{a}(x)-m(x)\} \qquad (\overline{a}\in \overline{\boldsymbol{R}}).$$

5) A modular m is said to be *uniformly finite,* if, for each $\xi>0$

$$\sup_{m(x)\leqq 1} m(\xi x)<\infty.$$

6) A modular m is said to be *uniformly increasing,* if

$$\lim_{\xi\to\infty}\frac{1}{\xi}\inf_{m(x)\geqq 1} m(\xi x)=\infty.$$

7) A modular m is said to be *reflexive,* if the relation:

$$m(a)=\sup_{\overline{x}\in\overline{\boldsymbol{R}}}\{\overline{x}(a)-\overline{m}(\overline{x})\} \qquad (a\in \boldsymbol{R}^m)$$

holds, where $\overline{m}$ is the conjugate modular of m. The reflexivity of m has been shown in [3, Theorem 39.3] and [5, §2.1].

8) The *norm* $|||\cdot|||$ is defined as $|||a|||=\inf_{m(\xi a)\leqq 1} 1/|\xi|$, and then we get $|||x||| \leqq \|x\|\leqq 2|||x|||$ for $x\in \boldsymbol{R}^m$.

$$\lim_{\xi\to\infty}\frac{1}{\xi}\inf_{m(x)\geq 1} m(\xi x)<C$$

Therefore, we have, for all sufficiently large ξ, $\inf\limits_{m(x)\geq 1} m(\xi x)<\xi C$, and there exist x_ξ such that

$$|||x_\xi|||\geqq 1 \text{ and } m(\xi x_\xi)\leqq \xi C,$$

since $|||x|||\leqq 1$ implies $m(x)\leqq |||x|||$. Accordingly we have

$$m(\xi x_\xi)/\xi|||x_\xi|||\leqq m(\xi x_\xi)/\xi\leqq C$$

for all sufficiently large ξ. This contradicts (5).

Next, we will prove the sufficiency. If (5) does not hold, then there exist x_n and C such that

(7) $$\lim_{n\to\infty}|||x_n|||=\infty,\ |||x_n|||\geqq 1,\ m(x_n)\leqq C|||x_n|||$$

and $$m(\xi_0 x_n)<\infty \text{ for some } \xi_0>1.$$

Moreover, we have $m(x_n/|||x_n|||)=1$, because if $m(x_n/|||x_n|||)<1$, then $m(\xi x_n/|||x_n|||)=\infty$ for all $\xi>1$ (by Lemma 2) and hence, putting $\xi=\xi_0|||x_n|||$, we get $m(\xi_0 x_n)=\infty$, namely, the contradiction of (7).

Therefore, for x_n in (7) we have

$$g(\xi_n)/\xi_n\leqq m(\xi_n x_n/|||x_n|||)/\xi_n=m(x_n)/|||x_n|||$$

where $$\xi_n=|||x_n||| \text{ and } g(\xi)=\inf_{m(x)\geq 1} m(\xi x).$$

And hence m is not uniformly increasing. Thus, on account of Lemma 1, Theorem 2 is completely proved.

Remark. Since, in L_M^*, the relations (2) and (3) imply the uniform increaseness of the modular m defined by (4), Theorem 2 is a extension of Theorem 1 to the modulared semi-ordered linear space.

References

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