# 6. *On the Images of Connected Pieces of Covering Surfaces. I*

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Let $w=f(z)$ be an analytic function in $|z|<1$. It is interesting to consider the distribution of zero-points of $f(z)-a=0$. Suppose $f(z)$ is of bounded type. Let $\{a_i\}$ be the set of $a$-points of $f(z)$. Then it is well known that $\sum_i G(z, a_i)<\infty$, where $G(z, a_i)$ is the Green's function of $|z|<1$. In the present paper we consider the distribution of the set $f^{-1}(C(\rho, w_0))$ in $|z|<1$, where $C(\rho, w_0)=E[w: |w-w_0|<\rho]$.

Let $R$ be a Riemann surface with positive boundary and let $R_n(n=0, 1, 2,\cdots)$ be its exhaustion with compact relative boundary $\partial R_n$. Let $G\subset G'$ be domains[1] in $R$, where $G$ and $G'$ may consist of at most enumerably infinite number of components. Let $w_n(z)$ be the least positive superharmonic function in $G'$ such that $w_n(z)\geqq 1$ on $G\cap(R-R_n)$. Put $w(B\cap G, z, G')=\lim_n w_n(z)$ and call it[2] H.M. of $(G\cap B)$. If there exists a number $n_0$ such that $D(\omega_n(z))<M<\infty$ for $n\geqq n_0$, where $\omega_n(z)$ is a harmonic function in $G'$ such that $\omega_n(z)=1$ on $G\cap R-R_n)$, $=0$ on $\partial G'$ and has M.D.I. (minimal Dirichlet integral), $\omega_n(z)\rightarrow$[2] in mean to $\omega(G\cap B, z, G')$ called C.P. of $(G\cap B)$. In case $G'=R$, we write $w(G\cap B, z)$ and if $G'=R-R_0$, we write $\omega(G\cap B, z)$ simply. Put $S(G, r)=E[z\in G: |z|=r]$.

Let $G$ be a domain (of one component) in $|z|<1$. If there exists no bounded harmonic function in $G$ vanishing on $\partial G$, i.e. $w(G\cap B, z, G)=0$, we say that $G$ is *almost compact*. Let $C(\rho, w_0)$ be a circle in the $w$-plane. Then $f^{-1}(C(\rho, w_0))$ is composed of at most enumerably infinite number of components (connected pieces) $g_1, g_2,\cdots$. If a domain $G$ is a subset of $\{g_i\}$, we call $G$ a D.G. (domain generated) of $f^{-1}(C(\rho,w_0))$. At first we shall prove by simple method the following

**Theorem 1.** *Let $w=f(z)$ be an analytic function in $|z|<1$ such that $|f(z)|\leqq M$.*

a) *Let $G$ be a* D.G. *of $f^{-1}(C(\rho, w_0))$ and let $G'$ be a* D.G. *of $f^{-1}(C(\rho' w_0))$ containing $G:\rho<\rho'$. Then $w(G\cap B, z)>0$ if and only if there exists at least one component $g'$ of $G'$ such that $w(G\cap B, z, g')>0$ for any $\rho'>\rho$.*

1) In the present paper we suppose the relative boundary of a domain consists of analytic curves clustering nowhere in $R$.

2) Z. Kuramochi: Potentials on Riemann surfaces: Journ. Sci. Hokkaido Univ. **14** (1962).

b) *Let $G$ be a domain in $|z|<1$. If $\overline{\lim\limits_{r\to1}}\,\mathrm{mes}\,(S(G,r))>0$, then $w(G\cap B,z)>0$.*

c) *Let $G$ be a D.G. of $f^{-1}(C(\rho,w_0))$ such that every component of $G$ is almost compact. Put $G''=G\cap f^{-1}(C(\rho'',w_0)):\rho''<\rho$. Then $\overline{\lim\limits_{r\to1}}\,\mathrm{mes}\,(S(G'',r))=0$ for any $\rho''<\rho$.*

Proof. a) Put $R=E[z:|z|<1]$ and $\underline{R}=E[w:|w|<M]$. Let ${}_{CG'}w(G\cap B,z)$ be the least positive superharmonic function in $R$ larger than $w(G\cap B,z)$ in $CG'$. Then ${}_{CG'}w(G\cap B,z)={}_{Cg'}w(G\cap B,z)$ in $g'$ for any component $g'$ of $G'$. Let $w(C(\rho,w_0),w)$ be H.M. of $C(\rho,w_0)$ (the least positive superharmonic function in $R$, $\geqq1$ on $C(\rho,w_0)$). Then since $R$ is of positive boundary, $w(C(\rho,w_0),w)\not\equiv1$ and by the maximum principle $w(C(\rho,w_0),w)\leqq N<1$ on $\partial C(\rho',w_0)$ and $\leqq N$ in $R-C(\rho',w_0)$. Since $w(C(\rho,w_0),w)$ can be considered in $R$, $w(G\cap B,z)\leqq w(C(\rho,w_0),w)$ $\leqq N$ in $R-f^{-1}(C(\rho',w_0))$. It is known that $w(G\cap B,z)$ is the least positive harmonic function in any domain $D$ among the functions with the same value as $w(G\cap B,z)$ on $\partial D$ such that $D\cap G=0$. Let $g'_i$ be a component of $G'$ and let $\hat{g}'_j$ be a component of $f^{-1}(C(\rho',w))-G'$. Then since $0=(\hat{g}'_j\cap G')\supset(\hat{g}'_j\cap G)$ and since $w(G\cap B,z)\leqq w(C(\rho,w_0),w)\leqq N$ on $\partial f^{-1}(C(\rho',w_0))$, $w(G\cap B,z)\leqq N$ in $\hat{g}'_j$. Next $\sup\limits_{z\in G'}{}_{CG'}w(G\cap B,z)\leqq\max w(G\cap B,z)$ on $\partial g'_i\leqq\max\limits_{W\in\partial C(\rho',w_0)}w(C(\rho,w_0),w)\leqq N$. Hence ${}_{CG'}w(G\cap B,z)\leqq w(G\cap B,z)\leqq N$ in $R-G'$ and ${}_{CG'}w(G\cap B,z)\leqq N$ in $G'$. Thus ${}_{CG'}w(G\cap B,z)$ $\leqq N$ in $R$. By $w(G\cap B,z)>0$ we have $\sup w(G\cap B,z)=1$[3] and $w(G\cap B,z)-{}_{CG'}w(G\cap B,z)>0$. Hence there exists at least one point $z_0$ such that $w(G\cap B,z_0)-{}_{CG'}w(G\cap B,z_0)>0$. Clearly $w(G\cap B,z)$ $={}_{CG'}w(G\cap B,z)$ outside of $G'$. Hence such $z_0\in G'$. Let $g'$ be the component of $G'$ containing $z_0$. Then $w(G\cap B,z)-{}_{Cg'}w(G\cap B,z)>0$. On the other hand, we have $w(G\cap B,z)-{}_{Cg'}w(G\cap B,z)=w(G\cap B,z,g')>0$.[4] Next suppose $w(G\cap B,z,g')>0$. Then by $g'\subset R$ and by the maximum principle $w(G\cap B,z)>0$. Thus we have a).

b) Let $w(r,z)$ be the least positive superharmonic function in $R$ and $\geqq1$ in $E[z\in G:|z|>r]$. Let $w'(r,z)$ be the least positive harmonic function in $|z|\leqq r$ and $\geqq1$ on $E[z\in G:|z|=r]$. Then $w(r,z)\geqq w'(r,z)$. Now $w'(r,z)=\dfrac{\mathrm{mes}\,S(G,r)}{2\pi r}$ at $z=0$. Since $w(B\cap G,z)$ $=\lim\limits_{r\to1}w(r,z)\geqq\overline{\lim\limits_{r\to1}}\,w(r,z)>0$ at $z=0$, we have $w(G\cap B,z)>0$.

c) Assume there exists a number $\rho''>\rho$ such that $\overline{\lim\limits_{r\to1}}\,\mathrm{mes}$ $S(G'',r)>0$: $G''$ is a D.G. of $f^{-1}(C(\rho'',w_0))$. Then by b) $w(G''\cap B,z)$ $>0$. Next by a) there exists at least one component $g$ of $G$ such that $w(G''\cap B,z,g)>0$. On the other hand, such $g$ is almost compact

3) See Theorem 2 of 2).
4) See p. 21 of 2).

and $w(G\cap B, z, g)$ must reduce to zero. This is a contradiction, whence we have c).

**Theorem 2.** *Let $w=f(z)$ be an analytic function in $|z|<1$ such that the spherical area of $R:E[z:|z|<1]$ by $f(z)$ is finite. Let $G$ be a* D.G. *of $f^{-1}(C(\rho, w_0))$ and let $G'$ be a* D.G. *of $f^{-1}(C(\rho', w_0)):\rho'>\rho$ containing $G$. Then*

a) *$\omega(G\cap B, z, R-R_0)>0$ if and only if there exists at least one component $g'$ of $G'$, a* D.G. *of $f^{-1}(C(\rho', w_0))$ such that $\omega(G\cap B, z, g')>0$ for any $\rho'>\rho$.*

b) *Let $G$ be a domain in $R$ (not necessarily of one component). If $\overline{\lim_{r\to 1}}\log.\mathrm{Cap}\, S(G, r)$ : logarithmic capacity: $>0$, $\omega(G\cap B, z)>0$.*

c) *Let $G$ be a* D.G. *of $f^{-1}(C(\rho, w_0))$ such that every component of $G$ is almost compact. Put $G''=G\cap f^{-1}(C(\rho'', w_0)):\rho''<\rho$. Then $\overline{\lim_{r\to 1}}\log.\mathrm{Cap}\, S(G'', r)=0$ for any $\rho''<\rho$.*

Let $R_n(n=0, 1,\cdots)$ be an exhaustion of $R$ with compact relative boundary $\partial R_n$ such that $R_1\supset R_0$ and dist $(\partial R_1, R_0)>0$. Let $\omega_n(z)$ be a harmonic function in $R-((R-R_n)\cap G)-R_0$ such that $\omega_n(z)=1$ on $(R-R_n)\cap G$, $=0$ on $\partial R_0$ and has M.D.I. ($\leqq M$), because dist $(\partial R_1, R_0)>0$ for $n\geqq 1$. Then $\omega_n(z)\to\omega(G\cap B, z, R-R_0)=\omega(G\cap B, z)$ in mean and also $D(\omega(G\cap B, z))\leqq M$. Let $U(w)$ be a continuous function in $\underline{R}$=whole $w$-sphere such that $U(w)=1$ on $C(\rho, w_0)$, harmonic in $C(\rho', w_0)-C(\rho, w_0)$ and$=0$ in $\underline{R}-C(\rho', w_0)$. Then $\left|\frac{\partial U(w)}{\partial u}\right|\leqq N$, $\left|\frac{\partial U(w)}{\partial v}\right|\leqq N$: $w=u+iv$. Put $U(z)=U(f^{-1}(w))$. Then $U(z)=1$ in $G$ and $=0$ on $\partial G'$ and $D(U(z))\leqq N^2\times$area of $G'$. Put $U_n'(z)=\min(U(z), \omega_n(z))$. Then $U_n'(z)=1$ on $G\cap(R-R_n)=0$ on $\partial G'+\partial R_0$ and $D(U_n'(z))\leqq D(\omega_n(z))+D(U(z))\leqq K<\infty$ for $n\geqq 1$. On the other hand, by the Dirichlet principle $D(U_n'(z))\geqq D(\omega_n'(z))$, where $\omega_n'(z)$ is a harmonic function in $(G'\cap(R-R_0))-(G\cap(R-R_n))$ such that $\omega_n'(z)=1$ on $G\cap(R-R_n)$, $=0$ on $\partial G'+\partial R_0$ and has M.D.I. Also by $(G'\cap(R-R_0))\subset(R-R_0)$ we have $D(\omega_n(z))\leqq D(\omega_n'(z))$. Hence by $D(\omega(G\cap B, z))>0$ and by the fact that $\omega_n'(z)\to\omega(G\cap B, z, G'\cap(R-R_0))$ in mean we have $D(\omega(G\cap B, z, G'\cap(R-R_0)))>0$. Next by $((R-R_0)\cap G')\subset G'$ we have by the maximum* principle[4] $\omega(G\cap B, z, G')\geqq\omega(G\cap B, z, G'\cap(R-R_0))>0$. Let $g_i'(i=1, 2,\cdots)$ be a component of $G'$. Then by $\underset{G'}{D}(\omega(G\cap B, z, G'))>0$, there exists at least one component $g'$ of $G'$ such that $\underset{g'}{D}(\omega(G\cap B, z, G'))>0$. Clearly in any component $g'$ $\omega(G\cap B, z, G')\equiv\omega(G\cap B, z, g')$. Hence there exists a component $g'$ of $G'$ such that $\omega(G\cap B, z, g')>0$. Next suppose $\omega(G\cap B, z, g')>0$. Then as above $0<\underset{g'}{D}(\omega(G\cap B, z, g'))\leqq D(\omega(G\cap B, z, g'\cap(R-R_0)))$ and by $((R-R_0)\cap g')\subset(R-R_0)$ we have $\omega(G\cap B, z)>0$.

b) Suppose $\overline{\lim_{r\to 1}}\log.\mathrm{Cap}\,(S(G, r))>0$. Then there exist a const.

$\delta>0$ and a sequence $r_1, r_2, \cdots$: $\lim_n r_n=1$ such that $\log.\mathrm{Cap}(S(G, r_n))>\delta$. Hence there exists a const. $\delta'$ such that $D(H(S(G, r_n), z))\geqq\delta'>0$, where $H(A, z)$ is a harmonic function in the whole $z$-plane$-D_0$ such that $H(A, z)=1$ on $A$, $=0$ on $\partial D_0$ and has M.D.I.$<\infty$, where $D_0$ is a disc such that $D_0\subset R_0$. Let $D_0^{r_n}$ be the mirror image of $D_0$ with respect to $|z|=r_n$. Then by the Dirichlet principle $D(\tilde{H}(S(G, r_n), z))\geqq\delta'$, where $\tilde{H}(A, z)$ is a harmonic function in the whole $z$-plane$-D_0-D_0^{r_n}$ such that $\tilde{H}(A, z)=1$ on $E[z: z\in A, |z|=r_n]$, $=0$ on $\partial D_0+\partial D_0^{r_n}$ and has M.D.I. Cleary $\tilde{H}(A, z)$ is symmetric with respect to $|z|=r_n$ and $\tilde{H}(A, z)$ has M.D.I. over $R'_n-D_0$: $R'_n=E[z: |z|<r_n]$. Hence $D_{R-R_0}(\omega(T(G, r_n)))\geqq D_{R-R_0}(\omega(S(G, r_n), z))\geqq D_{R'_n-D_0}(\tilde{H}(S(G, r_n), z)\geqq\frac{\delta_1}{2}$, where $\omega(A, z)$ is a harmonic function in $R-R_0$ such that $\omega(A, z)=1$ on $A$ and $=0$ on $\partial R_0$ and has M.D.I. over $R-R_0$ and $T(G, r)=E[z\in G: |z|>r]$, where $R=E[z: |z|<1]$. Now $\omega(T(G, r)z)\rightarrow\omega(G\cap B, z)$ in mean, as $r\rightarrow1$. Hence $\omega(G\cap B, z)>0$.

c) Assume there exists a number $\rho''<\rho$ such that $\overline{\lim_{r\to1}}\log.\mathrm{Cap}.\ S(G'', r)>0$. Then by b) $\omega(G''\cap B, z)>0$ and by $G''\subset G$ (by a)), there exists at least one component $g$ of $G$ such that $\omega(G''\cap B, z, g)>0$. On the other hand, since $g$ is almost compact, such function must be zero. Hence we have c) as c) of Theorem 1.

Let $f(z)$ be an analytic function in $R=E[z: |z|<1]$. If $f(z)$ has angular limits a.e. on $\Gamma=E[z: |z|=1]$, we say that $f(z)$ is of $F$-type. Clearly if $f(z)$ is of bounded type, $f(z)$ is of $F$-type.

**Theorem 3.** *Let $f(z)$ be of $F$-type. Let $F_\rho$ be the set on $\Gamma=E[z:|z|=1]$ such that $f(z)$ has angular limits in $C(\rho, w_0)$. Then $F_\rho$ is linearly measurable. Let $J(F_\rho, z)$ be the harmonic measure of $F_\rho$ in $R=E[|z|<1]$. Then*

a′) *$J(F_{\rho''}, z)\leqq w(\tilde{G}_\rho\cap B, z)\leqq J(F_{\rho'}, z)$, where $\tilde{G}_\rho=f^{-1}(C(\rho, w_0))$ for $\rho''<\rho<\rho'$.*

a) *Let $G$ be a D.G. of $f^{-1}(C(\rho, w_0))$ and let $G'$ be a D.G. of $f^{-1}(C(\rho', w_0))$ containing $G$. Then $w(G\cap B, z)>0$ if and only if there exists at least one component $g'$ of $G'$ such that $w(G\cap B, z, g')>0$ for any $\rho'>\rho$.*

b) *Let $G$ be a domain in $R$. If $\overline{\lim_{r\to1}}\mathrm{mes}(S(G, r))>0$, $w(G\cap B, z)>0$.*

c) *Let $G$ be a D.G. of $f^{-1}(C(\rho, w_0))$ such that every component of $G$ is almost compact. Put $G''=G\cap f^{-1}(C(\rho'', w_0)): \rho''>\rho$. Then $\overline{\lim_{r\to1}}\mathrm{mes}(S(G'', r))=0$ for any $\rho''<\rho$.*

d) Suppose the spherical image of $R$: $|z|<1$ is finite. *Let $G$ be a D.G. of $f^{-1}(C(\rho, w_0))$ and $G'$ be a D.G. of $f^{-1}(C(\rho', w_0))$ such that*

$G' \supset G$. *Then if* $\overline{\lim_{r\to 1}} \log.\mathrm{Cap}\,(S(G,r))>0$, $\underline{\lim}_{r\to 1}\,\mathrm{mes}\,(S(G',r))>0$ *for* $\rho'>\rho$.

Proof. a′) Let $\rho''<\rho$. Assume $\mathrm{mes}\,F_{\rho''}>0$. Then for any $\varepsilon>0$ there exists a closed set $E$ in $F_{\rho''}$ such that $\mathrm{mes}\,(F_{\rho''}-E)<\varepsilon$ and $f(z)$ converges uniformly on $E$. Hence for any $\rho^*$: $\rho''<\rho^*<\rho$ we can find a set $E'$ in $E$ and $\delta<1$ such that $\mathrm{mes}\,(F_{\rho''}-E')<2\varepsilon$ and $f(z)$ in $C(\rho^*, w)$ for $z$ in $D=E_{e^{i\theta}E'}\left[z\colon 1>|z|>\delta, \left|\arg\dfrac{z-e^{i\theta}}{e^{i\theta}}\right|<\dfrac{\pi}{4}\right]$, i.e. $\tilde{G}_\rho \supset D$. Now $D$ consists of a finite number of simply connected domains $\mathfrak{D}_i$. Suppose $\mathrm{mes}\,(\partial\mathfrak{D}_i\cap\Gamma)>0$. Then $w_n(z)\geqq U(z)$, where $U(z)$ is a harmonic function in $\mathfrak{D}_i$ such that $U(z)=0$ on $\partial\mathfrak{D}_i-\Gamma$ and $=1$ on $\partial\mathfrak{D}_i\cap\Gamma$ and $w_n(z)$ is the least positive superharmonic function in $|z|<1$ such that $w_n(z)\geqq 1$ in $\tilde{G}_\rho\cap(R-R_n)$, whence $w(\tilde{G}_\rho\cap B, z)\geqq U(z)$. Now $\partial\mathfrak{D}_i$ is rectifiable. Map $\mathfrak{D}_i$ onto $|\xi|<1$. Then $E'$ is mapped onto a set of positive measure on which $U(z)=1$ almost everywhere. Whence $U(z)=1$ a.e. on $E'$ and $U(z)\leqq w(\tilde{G}_\rho\cap B, z)$. Let $\varepsilon\to 0$. Then $w(\tilde{G}_\rho\cap B, z)=1$, a.e. on $F_{\rho^*}$ and $w(G_\rho\cap B, z)\geqq J(F_{\rho^*}, z)\geqq J(F_{\rho''}, z)$. If $\mathrm{mes}\,F_{\rho''}=0$, clearly $w(\tilde{G}_\rho\cap B, z)\geqq 0=J(F_{\rho''}, z)$. Next let $\rho'>\rho$ and $CF_{\rho^*}$ be the set on which $f(z)$ has angular limits not contained in $C(\rho^*, w_0)$: $\rho<\rho^*<\rho'$. Then $\tilde{G}_{\rho^*}=f^{-1}(C(\rho^*, w_0))$ does not tend to $CF_{\rho'}$ and it can be proved that $w(G\cap B, z)=0$ a.e. on $CF_{\rho'}$ as above. Hence $w(\tilde{G}_\rho\cap B, z)\leqq w(\tilde{G}_{\rho^*}\cap B, z)\leqq J(F_\rho, z)$.

a) Suppose $w(G\cap B, z)>0$. Then by a′) $w(G\cap B, z)\leqq w(\tilde{G}_\rho\cap B, z)\leqq J(F_{\rho^*}, z)$, where $\rho<\rho^*<\rho'$. Hence $E\subset F_{\rho^*}$ and $\mathrm{mes}\,E>0$, where $E$ is the set on $\Gamma$ on which $w(G\cap B, z)>0$. Then we can find a closed set $E'$ of positive measure in $E$ and a domain $D=E_{e^{i\theta}\in E'}\left[1>|z|>\delta, \left|\arg\dfrac{z-e^{i\theta}}{e^{i\theta}}\right|<\dfrac{\pi}{4}\right]$ such that $f(z)$ in $C(\rho', w_0)$ for $z\in D$. Then $\partial G_{\rho'}(\supset\partial G')$ does not tend to $D$, whence (Case 1) any component $\mathfrak{D}$ of $D$ is contained in a component $g'$ of $G'$ or (Case 2) $\mathfrak{D}$ is disjoint from any component $g'$ of $G'$. Case 1: Since ${}_{CG'}w(G\cap B, z)$ is the least positive superharmonic function in $\mathfrak{D}$ such that ${}_{CG'}w(G\cap B, z)=w(G\cap B, z)$ on $\partial\mathfrak{D}-\Gamma$, ${}_{CG'}w(G\cap B, z)=0$ a.e. on $\partial\mathfrak{D}\cap\Gamma$ Case 2: $0={}_{CG'}w(G\cap B, z)$ $(\leqq w(G\cap B, z)\leqq w(G'\cap B, z)=0$ a.e. on $\partial\mathfrak{D}\cap\Gamma$. Hence ${}_{CG'}w(G\cap B, z)=0$ a.e. on $\partial\mathfrak{D}\cap\Gamma$ and $=0$ a.e. on $\partial D\cap\Gamma$. Hence $0<w(G\cap B, z)-{}_{CG'}w(G\cap B, z)$. Thus there exists at least one component $g'$ of $G'$ such that $w(G\cap B, z, g')>0$.

b) and c) can be proved as Theorem 1.

d) Suppose $w(G\cap B, z)>0$. Then by a) $w(G\cap B, z)\leqq J(F_{\rho^*}, z)$: $\rho<\rho^*<\rho'$. Hence the set $E$ on $\Gamma$ on which $w(G\cap B, z)>0$ is contained in $F_{\rho^*}$ and $\mathrm{mes}\,E>0$. Hence as b) we can find a set $E'$ of positive measure in $E$ and a domain $D_{e^{i\theta}\in E'}=E\left[1>|z|>\delta,\ \left|\arg\frac{z-e^{i\theta}}{e^{i\theta}}\right|<\frac{\pi}{4}\right]$ such that $f(z)\in C(\rho^{**}, w_0)$: $\rho^*<\rho^{**}<\rho'$ for $z\in D$. Whence any component $\mathfrak{D}$ of $D$ is contained in a component $g'$ of $G'$ or disjoint from $G'$. If $\mathfrak{D}\cap G'=0$, $w(G\cap B, z)\leqq w(G'\cap B, z)=0$ a.e. on $\partial\mathfrak{D}\cap\Gamma$. Hence there exists at least one $\mathfrak{D}$ such that $\mathfrak{D}\subset g'$ of $G'$ and mes $(\partial\mathfrak{D}\cap\Gamma)>\delta>0$, whence $\delta<\varliminf_{r\to1} S(g', r)\leqq\varliminf_{r\to1} S(G' r)$. Next suppose $\varlimsup_{r\to1}$ log. Cap $S(G, r)>0$. Then by Theorem 2 there exists at least one component $g'$ of $G'$ such that $0<w(G\cap B, z, g')\leqq w(G\cap B, z)$. Hence we have by $\varlimsup_{r\to1}$ log. Cap $(S(G, r))>0$ $\varliminf_{r\to1} S(G', r)>0$.