25. On Theorems of Korovkin

By Hisashi CHODA and Marie ECHIGO
Department of Mathematics, Osaka Gakugei Daigaku
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1. In a recently published book [3], P. P. Korovkin established the following interesting theorems which are fundamental in his theory of approximation:

THEOREM 1. *If the two conditions*

(1) $$\sigma_n(1)\to 1, \quad as\ n\to\infty,$$

(2) $$\sigma_n(g)\to 0, \quad as\ n\to\infty,$$

where $a\leqq c\leqq b$ *and*

(3) $$g(x)=(x-c)^2,$$

are satisfied for the sequence of positive linear functionals σ_n *on the Banach space* $C[a, b]$ *of all continuous functions on* $[a, b]$, *then*

(4) $$\lim_{n\to\infty}\sigma_n(f)=f(c)$$

for any $f\in C[a, b]$.

THEOREM 2. *If the two conditions* (1) *and* (2) *are satisfied for the sequence of positive linear functionals* σ_n *on* $C[a, b]$ *and*

(5) $$g(x)=\sin^2\frac{x-c}{2},$$

where $a\leqq c\leqq b$, *then* (4) *is true for* $f\in C[a, b]$ *which has the period* 2π.

In this paper, we shall prove an abstract theorem which is a generalization of these theorems of Korovkin.

2. We shall introduce a few terms before we state our theorem. If a commutative Banach algebra A has an involution $x\to x^*$ satisfying $||xx^*||=||x||^2$ for any element x of A, then A will be called a *commutative B^*-algebra.* If a linear functional σ on a B^*-algebra A satisfies the condition that $\sigma(xx^*)\geqq 0$ for any element x of A, we shall say that σ is *positive.* It is well-known [4; p. 213] that a positive linear functional σ on a B^*-algebra A satisfies the inequality of Cauchy-Schwarz:

$$|\sigma(x^*y)|^2\leqq\sigma(|x|^2)\sigma(|y|^2)$$

for any $x, y\in A$, where $|x|=(x^*x)^{\frac{1}{2}}$. We shall call a positive linear functional σ a *state* whenever $\sigma(1)=1$ where 1 is the identity element of A. If a state χ of a commutative B^*-algebra is not expressible by a convex sum of two other states, χ will be called a *character.* It is also well-known [4; p. 229], that a character χ determines a maximal ideal M uniquely such as $M=\{x: \chi(x)=0\}$, and conversely that a maximal ideal M determines a character χ uniquely such that $\chi(x)$ coincides with the natural homomorphism of A onto A/M. Henceforth we shall call briefly that the character χ is *defined by* a maximal ideal M when χ corresponds with M in the above sense.

3. In what follows, we shall prove the following

THEOREM 3. *Let A be a commutative B^*-algebra with the identity element* 1, *M a principal maximal ideal generated by a, and χ the character defined by M. If the two conditions* (1) *and*

$$\sigma_n(|a|^2)\to 0, \quad as\ n\to\infty, \tag{6}$$

are satisfied for the sequence of positive linear functionals σ_n on A, then σ_n converges weakly to χ:*

$$\lim_{n\to\infty}\sigma_n(x)=\chi(x), \tag{7}$$

for any $x\in A$.

Proof. By the inequality of Cauchy-Schwarz and the assumption,

$$|\sigma_n(ax)|^2\leqq\sigma_n(|x|^2)\sigma_n(|a|^2)\leqq\|x\|^2\sigma_n(|a|^2)\to 0,$$

as $n\to\infty$, for any $x\in A$, whence $\sigma_n(ax)\to 0$. Since aA is dense in M by the assumption, $\sigma_n(y)$ converges to 0 for any $y\in M$. On the other hand, M is a hyperplane of A on which χ vanishes, whence each $x\in A$ is expressed in $x=\alpha 1+y$ by some $y\in M$ and a certain complex number α. Hence we have

$$\sigma_n(x)=\sigma_n(\alpha 1+y)=\alpha\sigma_n(1)+\sigma_n(y)\to\alpha=\chi(x),$$

as $n\to\infty$, for any $x\in A$.

4. At this end, we shall show that Theorem 3 implies Korovkin's theorems.

Let $A=C[a,b]$. Clearly A is a B^*-algebra with the identity element. Let M be the set of all elements of $C[a,b]$ vanishing at c and P the set of all polynomials. It is clear that M is a maximal ideal of A. Since P is dense in A, by a theorem of Yamabe (cf. [1; ch. 1, ex. 18, p. 55] or [2]) $P\frown M$ is dense in M. Now, let us define $h(x)=x-c$. Obviously h satisfies (6) since $g=h^2$ satisfies (2). Moreover, h generates M since each element of $P\frown M$ is divisible by h. Thus the all assumptions of Theorem 3 are satisfied and (7) implies (4), which proves Theorem 1.

Theorem 2 is also obtained similarly.

References

[1] N. Bourbaki: Espaces vectoriels topologiques, Actual. Sci. Ind., no. 1189, Paris (1953).

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[3] P. P. Korovkin: Linear Operators and Approximation Theory, Hindustan (1960).

[4] C. E. Rickart: General Theory of Banach Algebras, D. van Nostrand (1960).