23. *Inversive Semigroups. I*

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§1. **Introduction.** A semigroup S is called *inversive* if it satisfies the following

(C) (1) S has an idempotent, and the totality I of idempotents of S is a subband of S.
(2) For any element x of S, there exists an element x^* such that $xx^*=x^*x$ and $xx^*x=x$.

In this case, for any element x of S there exists one and only one element y such that $xy=yx$, $xyx=x$ and $yxy=y$. Such a y is called the *relative inverse* of x, and denoted by x^{-1}. Now, let $S(e)=\{x:xx^{-1}=e\}$ for each element e of I. Then, it is easy to see that each $S(e)$ is a subgroup of S and S is the class sum of all $S(e)$ (A. H. Clifford [1] has shown that a semigroup satisfying the condition (1) of (C), which is called a *semigroup admitting relative inverses*, is the class sum of subgroups). Therefore, inversive semigroups are not too far away from groups. Actually, as a special case, the author has proved in [4] that if I is a rectangular band then all $S(e)$ are isomorphic to each other and S is isomorphic to the direct product of an $S(e)$ and I. An inversive semigroup in which the totality of idempotents is a rectangular subband is called an *(R)-inversive semigroup.*

Now, it is clear that any (R)-inversive semigroup satisfies the following

(C. 1) If $xx^{-1}=e$ and if f is an idempotent such that $f\leqq e$ (i.e. $fe=ef=f$), then $fx=xf$.

However, an inversive semigroup satisfying the condition (C. 1) is not necessarily (R)-inversive. By a *strictly inversive semigroup,* we shall mean an inversive semigroup satisfying the condition (C. 1). The main purpose of this paper is to present a structure theorem for strictly inversive semigroups, and some relevant matters. The proofs are omitted and will be given in detail elsewhere.[1]

§2. **The structure of strictly inversive semigroups.** Let G be a semigroup. If there exist a band Ω and a collection $\{G_\alpha:\alpha\in\Omega\}$ of subsemigroups of type $\mathfrak{T}$ such that

(i) $G=\bigcup\{\alpha_\alpha:\alpha\in\Omega\}$,

(ii) $G_\beta\cap G_\gamma=\phi$ for $\beta\neq\gamma$

and (iii) $G_\beta G_\gamma\subset G_{\beta\gamma}$.

1) This is an abstract of the paper which will appear elsewhere.

then we shall say that G is a band Ω of semigroups G_α of type $\mathfrak{T}$.

In this sense, the following results follow from the papers [1], [2] of A. H. Clifford:

(I) A semigroup admitting relative inverses is a semilattice of completely simple semigroups.

(II) A semigroup G admitting relative inverses is a band of groups if and only if $Gba=Gba^2$ and $abG=a^2bG$ for any elements a, b of G.

Next, we shall define some special inversive semigroups. Let S be an inversive semigroup, and I the band consisting of all idempotents of S. Then, S is called *normal, left normal, right normal, commutative, rectangular, left singular, right singular* or *trivial* respectively, if it satisfies the following corresponding identity

(N) $xyzw=xzyw$, $(L.N)$ $xyz=xzy$, $(R.N)$ $xyz=yxz$, (C) $xy=yx$,

(R) $xyz=xz$, $(L.S)$ $xy=x$, $(R.S)$ $xy=y$ or (T) $x=y$.

Moreover, S is called (N)-*inversive*, $(L.N)$-*inversive*, $(R.N)$-*inversive*, (C)-*inversive*, (R)-*inversive*, $(L.S)$-*inversive*, $(R.S)$-*inversive* or (T)-*inversive* respectively, if I satisfies the above-mentioned corresponding identity (N), $(L.N)$, $(R.N)$, (C), (R), $(L.S)$, $(R.S)$ or (T).

Remark. The structure of bands satisfying one of the above-mentioned identities has been completely determined by the author [4] and N. Kimura and author [3]. Further, it is also clear that any (T)-inversive semigroup is a group and any trivial inversive semigroup is a trivial band.

Under these definitions, we have the following theorem, which is a special case of the above-mentioned result (I).

Theorem 1. *An inversive semigroup S is expressible as a semilattice of (R)-inversive semigroups. That is, there exist a semilattice Γ and a collection $\{S_\gamma : \gamma \in \Gamma\}$ of (R)-inversive subsemigroups such that*

$$\text{(i)}\quad S=\bigcup\{S_\gamma : \gamma\in\Gamma\},$$

$$\text{(ii)}\quad S_\alpha \cap S_\beta = \phi \quad \textit{for } \alpha \neq \beta$$

and

$$\text{(iii)}\quad S_\alpha S_\beta \subset S_{\alpha\beta}.$$

Further, Γ is determined uniquely up to isomorphism, and accordingly so are the S_γ.

Now, for a strictly inversive semigroup we have

Lemma 1. *If $aa^{-1}=e$ and $bb^{-1}=f$, then $(ab)^{-1}=eb^{-1}a^{-1}f$.*

Lemma 2. *If $aa^{-1}=e$ and $bb^{-1}=f$, then $(ab)(ab)^{-1}=ef$.*

Using these two lemmas, we obtain the following theorem as a special case of the above-mentioned result (II).

Theorem 2. *Let S be an inversive semigroup, and I the band consisting of all idempotents of S. Then, S is expressible as a band of groups if and only if S is strictly inversive. Further, in this case such a decomposition is uniquely determined, and it is the de-*

composition such that

$$\text{(i)}\quad S=\bigcup\{S(e):e\in I\},$$
$$\text{(ii)}\quad S(f)\cap S(h)=\phi \quad \textit{for } f\neq h,\ f,h\in I$$
and
$$\text{(iii)}\quad S(f)S(h)\subset S(fh),$$
where $S(e)=\{x:xx^{-1}=e\}$ *for every* e *of* I.

We shall call Γ in Theorem 1 the *strcuture semilattice* of S and S_γ the *γ-kernel* of S. Also in this case we write $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ and call it the *structure decomposition* of S.

Remark. If S is an inversive semigroup, then the band I of all idempotents of S is also inversive. In this case, it is easy to see that S and I have the same structure semilattice.

Next, we have

Lemma 3. *A (N)-inversive semigroup is strictly inversive.*

Particularly a (C)-inversive semigroup is strictly inversive, because a semilattice satisfies normality. Therefore, by Theorem 2 a (C)-inversive semigroup is expressible as a band of groups.

Moreover, for (C)-inversive semigroups Theorem 2 is sharpened as follows:

Theorem 3. *A semigroup is expressible as a semilattice of groups if and only if it is (C)-inversive (see also A. H. Clifford* [2]).

Therefore, the problem of constructing all possible (C)-inversive semigroups is reduced to the problem of finding all possible compound semigroups of $\{S_\gamma:\gamma\in\Gamma\}$ by Γ for a given semilattice Γ and a given collection $\{S_\gamma:\gamma\in\Gamma\}$ of groups S_γ. This problem was completely solved by A. H. Clifford [1] and the author [5].[2]

Next, we shall introduce a special kind of subdirect product, which is called the *spined product.*

Let S_1, S_2 be inversive semigroups having Γ as their structure semilattices, and $S_1\sim\sum\{S_1^\gamma:\gamma\in\Gamma\}$ and $S_2\sim\sum\{S_2^\gamma:\gamma\in\Gamma\}$ be the structure decompositions of S_1 and S_2. Then, the set $S=\bigcup\{S_1^\gamma\times S_2^\gamma:\gamma\in\Gamma)$ becomes a subdirect product of S_1 and S_2.[3] Such an S is called the spined product of S_1 and S_2 with respect to Γ, and denoted by $S_1\bowtie S_2(\Gamma)$. We sometimes omit (Γ), if there is no confusion.

Under this definition, we have the following main theorem.

Theorem 4. (Structure theorem.) *Let S be a strictly inversive semigroup having Γ as its structure semilattice. Let I be the band consisting of all idempotents of S. Then, S is isomorphic to $C\bowtie I(\Gamma)$ for some (C)-inversive semigroup C having Γ as its structure semilattice. Conversely, if C and I are a (C)-inversive semigroup and a band having Γ as their structure semilattices, then $C\quad I(\Gamma)$ is a strictly inversive semigroup. In other words, a semigroup is iso-*

2) For definition of a compound semigroup, see the author [5].

3) For definition of a subdirect product, see G. Birkhoff: Lattice Theory, p. 91.

morphic to the spined product of a (C)-inversive semigroup and a band if and only if it is strictly inversive.

From Theorem 4, the problem of determining the structure of strictly inversive semigroups is reduced to the problems of determining the structures of (C)-inversive semigroups and bands. As stated above, the former was completely solved by Theorem 3, A. H. Clifford [1] and the author [5]. The latter was also partially solved by several papers.

§3. Applications for (N)-inversive semigroups. Since a P-inversive semigroup, where P is (N), ($L.N$), ($R.N$), (C), (R), ($L.S$), ($R.S$) or (T), is (N)-inversive and since any (N)-inversive semigroup is strictly inversive, we have the following corollaries for P-inversive semigroups as special cases of Theorem 4.

Corollary 1. *A semigroup is isomorphic to the spined product of a* (C)*-inversive semigroup and a normal* (*left normal, right normal*) *band if and only if it is* (N)-(($L.N$)-, ($R.N$)-) *inversive.*

Corollary 2. *A semigroup is isomorphic to the direct product of a group and a rectangular* (*left singular, right singular*) *band if and only if it is* (R)- (($L.S$)-, ($R.S$)-) *inversive* (*see also the author* [4]).

Corollary 3. *A semigroup is isomorphic to the spined product of a commutative inversive semigroup and a normal* (*left normal, right normal*) *band if and only if it is a normal* (*left normal, right normal*) *inversive semigroup.*

Remark. For construction of commutative inversive semigroups, see the author [5]; for that of normal, left normal or right normal bands, see N. Kimura and the author [3].

Corollary 4. *A rectangular* (*left singular, right singular*) *inversive semigroup is a rectangular* (*left singular, right singular*) *band.*

References

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