147. On the Point Spectrum of the Schrödinger Operator

By Sigeru MIZOHATA and Kiyoshi MOCHIZUKI

(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1963)

1. Introduction. Let us consider the Schrödinger operator defined in R^3

$$L=\sum_{j=1}^{3}\left(\frac{1}{i}\frac{\partial}{\partial x_j}-b_j(x)\right)^2+q(x) \tag{1.1}$$

$$\equiv-\Delta+2i\sum b_j\frac{\partial}{\partial x_j}+i\sum\frac{\partial b_j}{\partial x_j}+c(x),$$

where $b_j(x)$ and $q(x)$ are real-valued. Our purpose is to show that, under certain conditions on b_j and q, the point spectrum of the operator L is finite.

Let us assume[1)]

(C_1) $\quad b_j(x)\in\mathcal{B}^1(R^3),\quad c(x)\in\mathcal{E}^0(Co),\quad |c(x)|\leq\dfrac{\text{const}}{|x|^{1.5-\varepsilon}}+\text{const},\quad \varepsilon>0.$

Under this assumption, it is easy to see

Lemma 1.1. *The operator L has a unique self-adjoint extension A, and $\mathcal{D}(A)=\mathcal{D}^2_{L^2}$, moreover we have*

$$\|u(x)\|_{\mathcal{D}^2_{L^2}}\leq C(\Lambda)\|u\|_{L^2} \tag{1.2}$$

for any eigenfunction $(\lambda-A)u=0$ for $\lambda\leq\Lambda$, Λ being arbitrary positive number.

In section 2, we require more stringent condition:

(C_2) $\quad b_j(x)\in\mathcal{E}^2(R^3);\ b_j(x),\ \ |x|\dfrac{\partial b_j}{\partial x_i}(x)$ are bounded; $c(x)\in\mathcal{E}^1(Co)$;

$$|x|\cdot\left|\frac{\partial c}{\partial x_i}(x)\right|\leq\frac{\text{const}}{|x|^{1.5-\varepsilon}}+\text{const},\quad \varepsilon>0.$$

Then, under the assumptions (C_1) and (C_2), we have

Lemma 1.2. *Let $u(x)\in\mathcal{D}^2_{L^2}$ be a solution of $Au=\lambda u$, λ real. We have $u(x)\in\mathcal{E}^3_{L^2(\mathrm{loc})}(Co)$. Moreover, in a neighbourhood of the origin, we have*

$$|u(x)|\leq\text{const},\quad |u_{x_i}(x)|\leq\frac{\text{const}}{|x|^{0.5-\varepsilon}},\quad |u_{x_ix_j}(x)|\leq\frac{\text{const}}{|x|^2}.$$

2. Upper boundedness of the eigenvalues.

Theorem 1. *Under the assumptions* (C_1), (C_2), *there exists a* $\lambda_0>0$

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1) In this note, we used the notations of L. Schwartz in his treatise (Théorie des Distributions). Let us explain these briefly: $f(x)\in\mathcal{B}^m$, if $f(x)$ has continuous bounded derivatives up to order m. $f(x)\in\mathcal{E}^m(\Omega)$, if f is merely continuously differentiable in Ω up to order m. $\mathcal{D}^m_{L^2}$ is the space of all functions such that $D^\nu f\in L^2(R^n)$, $|\nu|\leq m$, $\|f\|^2_{\mathcal{D}^m_{L^2}}=\sum_{|\nu|\leq m}\|D^\nu f\|^2_{L^2}$. $\mathcal{E}^m_{L^2}(\Omega)$ is the space of all functions such that $D^\nu f(x)\in L^2(\Omega)$, $|\nu|\leq m$, with the norm: $(\sum_{|\nu|\leq m}\|D^\nu f\|^2_{L^2(\Omega)})^{\frac{1}{2}}$. $f\in\mathcal{E}^m_{L^2(\mathrm{loc})}(\Omega)$, if $\alpha f\in\mathcal{E}^m_{L^2}(\Omega)$, for all $\alpha(x)\in\mathcal{D}(\Omega)$.

such that, for $\lambda\in[\lambda_0,\infty)$ there exists no eigenvalue of A.

Proof. We follow the Wienholtz work ([3]). Let us assume, for the moment, $u(x)\in\mathcal{E}^3(Co)$. Let us start from the identity:

$$(*)\quad (n+2)\{(-\Delta u)\bar{u}+(-\Delta\bar{u})u\}=-n\sum_i\Big(\frac{\partial}{\partial x_i}|u|^2\Big)_{x_i}+2\sum_{i,k}(x_i|u_{x_k}|^2)_{x_i}$$
$$-2\sum_{i,k}(x_iu_{x_ix_k}\bar{u}+x_i\bar{u}_{x_ix_k}u)_{x_k}+2|x|\Big\{u\frac{\partial}{\partial|x|}\Delta\bar{u}+\bar{u}\frac{\partial}{\partial|x|}\Delta u\Big\}\ \text{in } R^n.$$

Let $u(x)\in\mathcal{D}^2_{L^2}$ be a solution of $Au=\lambda u$. Taking into account of (1.1),

$$2|x|\Big\{u\frac{\partial}{\partial|x|}\Delta\bar{u}+\bar{u}\frac{\partial}{\partial|x|}\Delta u\Big\}=2n(\lambda-c(x))|u|^2+I+J-4iK,^{2)}\ \text{where}$$
$$I=2i|x|\Big\{\sum_j\frac{\partial b_j}{\partial x_j}\Big(\bar{u}\frac{\partial}{\partial|x|}u-u\frac{\partial}{\partial|x|}\bar{u}\Big)\Big\},$$
$$J=2|x|\frac{\partial}{\partial|x|}c(x)\cdot|u|^2+2\sum_i\{(c(x)-\lambda)u\bar{u}\}_{x_i}$$
$$K=\sum_{i,j}(x_ib_ju\bar{u}_{x_j})_{x_i}-(x_ib_j\bar{u}u_{x_i})_{x_j}+\sum_{i,j}x_i\frac{\partial b_j}{\partial x_i}(u\bar{u}_{x_j}-\bar{u}u_{x_i})$$
$$+\sum_{i,j}\frac{\partial}{\partial x_j}(x_ib_j)u_{x_j}\bar{u}-\sum_{i,j}\frac{\partial}{\partial x_i}(x_ib_j)u\bar{u}_{x_j}.$$

Now, taking into account of (1.1),

$$\int_{\varepsilon\le|x|\le r}(n+2)\{(-\Delta u)\bar{u}+(-\Delta\bar{u})u\}-2n(\lambda-c(x))|u|^2\,dx$$
$$\ge 2\int(\lambda-c(x))|u|^2\,dx+2\int|\text{grad}\,u|^2\,dx-\text{const}\int|u|\,|\text{grad}\,u|\,dx$$
$$-\text{const}\Big(\int_{|x|=r}|u|\,|\text{grad}\,u|\,dS+\int_{|x|=\varepsilon}|u|\,|\text{grad}\,u|\,dS\Big),$$

where const means a constant independent of $u(x)$, r and λ. This convention will be made hereafter. The integration is taken over $\{x;\varepsilon\le|x|\le r\}$. Next, we integrate (the second member of (*)) $-2n(\lambda-c(x))|u|^2$ on the domain $\varepsilon\le|x|\le r$. Taking into account of Lemma 1.2, this integral is estimated by the following form:

$$\text{const}\,(1+r+\lambda r)\int_{|x|=r}|u|^2+|\text{grad}\,u|^2+\sum_{i,j}|u_{x_ix_j}|^2\,dS$$
$$+\text{const}\int_{|x|\le r}|u|\,|\text{grad}\,u|\,dx+2\int_{|x|\le r}|x|\Big|\frac{\partial}{\partial|x|}c(x)\Big|\,|u|^2\,dx.$$

Therefore we have the inequality

$$(2.2)\quad 2\int_{|x|\le r}(\lambda-c(x))|u|^2\,dx+2\int_{|x|\le r}|\text{grad}\,u|^2\,dx-\text{const}\int_{|x|\le r}|u|\,|\text{grad}\,u|\,dx-$$

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2) $I=2|x|\Big\{u\frac{\partial}{\partial|x|}\Big(-i\sum\frac{\partial b_j}{\partial x_j}\bar{u}\Big)+\bar{u}\frac{\partial}{\partial|x|}\Big(i\sum\frac{\partial b_j}{\partial x_j}u\Big)\Big\}$,

$J=2|x|\Big\{u\frac{\partial}{\partial|x|}(c(x)-\lambda)\bar{u}+\bar{u}\frac{\partial}{\partial|x|}(c(x)-\lambda)u\Big\}-2n(\lambda-c(x))|u|^2$,

$K=u|x|\sum\frac{\partial}{\partial|x|}(b_j\bar{u}_{x_j})-\bar{u}|x|\sum\frac{\partial}{\partial|x|}(b_ju_{x_j})$.

$$-2\int_{|x|\le r}|x|\left|\frac{\partial}{\partial|x|}c(x)\right||u|^2\,dx$$
$$\le \text{const}\,(1+r+\lambda r)\int_{|x|=r}|u|^2+|\operatorname{grad} u|^2+\sum|u_{x_ix_j}|^2\,dS.$$

Up to now, we assumed $u(x)\in\mathcal{E}^3(Co)$. We can remove this assumption. Take a mollifier $\varphi_\delta(x)$, and consider $u_\delta=\varphi_\delta * u(x)$. u_δ satisfies

(2.3) $Au_\delta+C_\delta u=\lambda u_\delta$, where $C_\delta=[\varphi_\delta *, B]$, $B=2i\sum b_j\dfrac{\partial}{\partial x_j}+i\sum\dfrac{\partial b_i}{\partial x_j}+c(x)$.

Since $u(x)\in\mathcal{E}^3_{L^2(\mathrm{loc})}(Co)$ (Lemma 1.2), $u_\delta\to u$ in $\mathcal{E}^3_{L^2(\mathrm{loc})}(Co)$, and that we know $\displaystyle\int_{\varepsilon\le|x|\le r}\left|\frac{\partial}{\partial|x|}C_\delta u\right|^2 dx\to 0$, as $\delta\to 0$. This shows that, by the passage to the limit, the above reasoning is also true.

Finally we have, taking into account of the condition (C_1),

(2.4) $\displaystyle\int_{|x|\le r}|x|\left|\frac{\partial}{\partial|x|}c(x)\right||u|^2\,dx\le c_0\sqrt{\delta}\int_{|x|\le r}|\operatorname{grad} u|^2\,dx+c(\delta)\int_{|x|\le r}|u|^2\,dx,$[3)]

where δ can be taken arbitrarily small.

Finally, taking into account of (C_1) and of $(**)$ of the footnote 3), if we choose $\lambda_0>0$ sufficiently large,

(2.5) $\displaystyle 2(\lambda-\lambda_0)\int_{|x|\le r}|u|^2\,dx\le\text{const}\,(1+r+\lambda r)\int_{|x|=r}|u|^2+|\operatorname{grad} u|^2+\sum|u_{x_ix_j}|^2 dS.$

Dividing both sides by r, and integrating in r from $a(>0)$ to R, we have

$$2(\lambda-\lambda_0)\int_{|x|\le a}|u|^2\,dx\cdot\log\frac{R}{a}\le\text{const}\int_{a\le|x|\le R}|u|^2+|\operatorname{grad} u|^2+\sum|u_{x_ix_j}|^2\,dx.$$

Since $u(x)\in\mathcal{D}^2_{L^2}$, the right hand side tends to a finite limit when $R\to+\infty$, hence $u(x)\equiv 0$ for $|x|\le a$. Since a is arbitrary, we have $u(x)\equiv 0$.

3. Finiteness of positive eigenvalues. We impose the following conditions on the behavior of b_j and c at infinity.

(C_3) $b_j(x)=b_j^0+\bar{b}_j(x)$, b_j^0 being real; $\bar{b}_j(x)$, $\dfrac{\partial\bar{b}_j}{\partial x_i}(x)$, $q(x)=O\left(\dfrac{1}{|x|^{2+\varepsilon}}\right)$, $\varepsilon>\dfrac{1}{2}$, as $|x|\to+\infty$.

We want to prove

Theorem 2. *Under the assumptions* (C_1), (C_3), *for any* $\Lambda>0$, *there exists at most a finite number of eigenvalues of the operator* A *in* $[0,\Lambda]$. *Here the number is counted with multiplicity.*

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3) In fact

$$(**)\quad\int_{|x|\le r}\frac{|u(x)|^2}{|x|^{1.5-\varepsilon}}dx=\int_{|x|\le\delta}\cdots+\int_{\delta\le|x|\le r}\cdots\le\sqrt{\delta}\int_{|x|\le\delta}\frac{|u(x)|^2}{|x|^2}dx+\frac{1}{\delta^{1.5-\delta}}\int_{|x|\le r}|u(x)|^2\,dx$$
$$\le 4\sqrt{\delta}\int_{|x|\le r}|\operatorname{grad} u|^2\,dx+c(\delta)\int_{|x|\le r}|u(x)|^2\,dx.$$

Since

$$e^{-i(b_j^0x)}\sum_j\left(\frac{1}{i}\frac{\partial}{\partial x_j}-b_j(x)\right)^2 u=\sum_j\left(\frac{1}{i}\frac{\partial}{\partial x_j}-\bar{b}_j(x)\right)^2(e^{-i(b_j^0x)}u),$$

in order to prove Theorem 2, it is enough to assume that $b_j(x)$ themselves satisfy (C_3), therefore $c(x)=\sum b_j^2(x)+q(x)$ satisfies the same condition as $q(x)$ in (C_3). So we assume

$$(C_4)\qquad b_j,\ \frac{\partial b_j}{\partial x_i},\ c(x)=O\Big(\frac{1}{|x|^{2+\varepsilon}}\Big),\quad \varepsilon>\frac{1}{2}.$$

It is easy to see that, if $u(x)\in\mathcal{D}(A)=\mathcal{D}^2_{L^2}$ satisfies $Au=\lambda^2 u$, $\lambda\geq 0$, we have

$$(3.1)\quad u(x)=-\frac{1}{4\pi}\int\frac{e^{i\lambda|x-y|}}{|x-y|}\Big\{2i\sum b_j(y)\frac{\partial}{\partial y_j}+i\sum\frac{\partial b_j}{\partial y_j}(y)+c(y)\Big\}u(y)dy.$$

Now we prove the following lemma due essentially to Povzner ([2]):

Lemma 3.1. *Let us consider the function*

$$(3.2)\qquad \psi(x)=\int\frac{e^{i\lambda|x-y|}}{|x-y|}a(y)u(y)dy,\quad u(x)\in L^2,$$

$a(x)$ being bounded, and when $|x|\to+\infty$, $a(x)=O\Big(\frac{1}{|x|^{2+\varepsilon}}\Big)$, $\varepsilon>\frac{1}{2}$. Then

$$(3.3)\quad \psi(x)=\frac{e^{i\lambda|x|}}{|x|}\int e^{-i\lambda(\tilde{x},y)}a(y)u(y)dy+\psi_1(x)\equiv\psi_0(x)+\psi_1(x),\ \textit{where}\ \tilde{x}=\frac{x}{|x|},$$

moreover

$$(3.4)\quad |\psi_1(x)|\leq\frac{\text{const}}{(1+|x|)^{1.5+\delta}}\|u\|_{L^2},\ \textit{where}\ \delta=\frac{\varepsilon}{2}-\frac{1}{4},\ \textit{const is independent}$$

of λ. In particular, if $\psi(x)\in L^2$, then $\psi_0(x)\equiv 0$.

Proof. Let us write

$$\psi(x)=\int_{|y|\leq\rho}\frac{e^{i\lambda|x-y|}}{|x-y|}a(y)u(y)dy+\int_{|y|\geq\rho}\cdots dy.$$

$$|\text{second term}|\leq\int_{|y|\geq\rho}\frac{|a(y)|}{|x-y|}|u(y)|dy\leq c\int_{|y|\geq\rho}\frac{|u(y)|}{|x-y|\cdot|y|^{2+\varepsilon}}dy$$

$$\leq c\|u\|\Big(\int_{|y|\geq\rho}\frac{dy}{|x-y|^2|y|^{4+2\varepsilon}}\Big)^{\frac{1}{2}}\leq c'\|u\|\frac{1}{|x|\cdot\rho^{\frac{1}{2}+\varepsilon}}.^{4)}$$

Put

$$(3.5)\qquad \rho=|x|^{\frac{1}{2}},$$

we have

$$|\text{second term}|\leq\frac{1}{|x|^{1.5+\left(\frac{\varepsilon}{2}-\frac{1}{4}\right)}}c'\|u\|.$$

Concerning the first term,

$$\frac{e^{i\lambda|x-y|}}{|x-y|}=\frac{e^{i\lambda|x|}}{|x|}e^{-i\lambda(\tilde{x},y)}e^{i\lambda|x|O(\tau^2)}(1+O(\tau))$$

4) In fact

$$\int_{|y|\geq\rho}\frac{dy}{|x-y|^2|y|^{3+h}}\leq\frac{\text{const}}{|x|^2\rho^h},\ h>0.\ \text{See [1], p. 20.}$$

$$=\frac{e^{i\lambda|x|}}{|x|}e^{-i\lambda(\tilde{x},y)}(1+|x|O(\tau^2))(1+O(\tau)),\ \text{where } \tau=\frac{|y|}{|x|}.$$

Put
$$g(x)=\frac{e^{i\lambda|x|}}{|x|}\int_{|y|\le\rho}e^{-i\lambda(\tilde{x},y)}|x|O(\tau^2)a(y)u(y)dy$$

we have

$$|g(x)|\le\frac{\text{const}}{|x|^2}\int_{|y|\le\rho}\frac{|y|^2}{(1+|y|)^{2+\varepsilon}}|u(y)|dy\le\frac{\text{const}}{|x|^2}\rho^{\frac{3}{2}-\varepsilon}||u||\le\frac{\text{const}}{|x|^{1.5+\left(\frac{\varepsilon}{2}-\frac{1}{4}\right)}}||u||.$$

Concerning the other terms, we have easier estimates. Finally

$$\left|\frac{e^{i\lambda|x|}}{|x|}\int_{|y|\ge\rho}e^{-i\lambda(\tilde{x},y)}a(y)u(y)dy\right|\le\frac{\text{const}}{|x|\cdot\rho^{\frac{1}{2}+\varepsilon}}||u||=\frac{\text{const}}{|x|^{1.5+\left(\frac{\varepsilon}{2}-\frac{1}{4}\right)}}||u||.$$

Remark. We can apply this lemma to the integrands in (3.1). Concerning the term $c(y)u(y)$, since $c(x)$ is not bounded, we take the following precaution:

$$\int_{|y|\le1}|c(y)u(y)|^2dy\le c||u||_{\mathcal{D}^2_{L^2}}\le cC(\Lambda)||u||_{L^2}\ \text{(Lemma 1.1).}$$

Finally we see that the lemma is also true for $\lambda=0$.

Lemma 3.2. (Equi-continuity). *The eigenfunctions* $u(x)\in\mathcal{D}^2_{L^2}$ *corresponding to* $\lambda\in[0,\Lambda]$ *are uniformly bounded and equicontinuous, provided that* $||u||_{L^2}=1$.

Proof. Uniform boundedness is an immediate consequence of Lemma 1.1. To show the equi-continuity, it is enough to remark that

$$\varphi(x)=\int_{|y|\le R}\frac{e^{i\lambda|x-y|}}{|x-y|}v(y)dy,\ \lambda\in[0,\Lambda],$$

satisfies

$$|\varphi(x)-\varphi(x')|\le C_R|x-x'|^{\frac{1}{2}}||v||_{L^2},$$

and also the above remark.

Lemmas 3.1 and 3.2 show that the set of all eigenfunctions $u(x)\in\mathcal{D}^2_{L^2}$ corresponding to $\lambda\in[0,\Lambda]$, $||u(x)||_{L^2}=1$ forms a compact set in L^2. This proves Theorem 2.

Final remark. If we apply a recent work of Birman to (1.1), we can affirm the finiteness of the negative discrete spectrum. Namely, let us assume

$$(\mathrm{C}_5)\qquad q(x),\quad b^2(x)\equiv\sum_j\bar{b}_j(x)^2=O\left(\frac{1}{|x|^{2+\varepsilon}}\right),\quad \varepsilon>0,\quad \text{for } |x|\to+\infty,$$

where $b_j(x)=b^0_j+\bar{b}_j(x)$, b^0_j are real constants, then under the assumptions (C_1) and (C_5), the negative eigenvalues are finite. Let us remark, above all, that as in Theorem 2, we can assume instead of $\bar{b}_j(x)$, $b_j(x)$ themselves satisfy (C_5). Following the notation of [4], let us write (1.1) under the form

$$C=\left\{\sum_j\left(\frac{1}{i}\frac{\partial}{\partial x_j}-b_j(x)\right)^2+b^2(x)+q^+(x)\right\}-\{b^2(x)+q^-(x)\}\equiv A-B,$$

$D(A)=D(B)=\mathcal{D}(R^3)$. Now, let H_A be the completion of $D(A)$ by the metric $(Au, u)^{\frac{1}{2}}$. It is easy to see that H_A is the same as the completion of $D(A)$ by the metric $\|\operatorname{grad} u\|_{L^2}$. Since the form $B[u, u]$ is completely continuous in H_A, we can apply Theorem 1.3 of [4].

References

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